Towards the Atiyah Conjecture for Link Groups and their Extensions

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultäten
der Georg-August-Universität zu Göttingen

vorgelegt von
Inga Christine Blomer, geb. Kümpel
aus Kiel

Göttingen 2007

D7
Referent: Prof. Thomas Schick
Korreferent: Prof. Ulrich Stuhler
Tag der mündlichen Prüfung: 11. Juli 2007

Contents

Acknowledgements

I would like to thank my advisor Prof. Thomas Schick for his excellent guidance and advice. Moreover I would like to thank Prof. John Labute. The discussions we had at Oberwolfach as well as the e-mail correspondence were very fruitful and interesting. Special thanks also to the DFG-Graduate School ”Gruppen und Geometrie” in Göttingen for generous funding and a good research environment. Additional funding was given by the Prof. Rhein Foundation, whom I thank as well. I would also like to thank the Department of Mathematics at Nice University in France, in particular Prof. Patras and Prof. Berger, and the German Academic Exchange Agency (DAAD), who funded the visit. Special thanks to the Mathematical Research Institute Oberwolfach. It was a great experience to attend a workshop on pro-p groups.
Finally I would like to thank Valentin.

Chapter 1

Introduction

The Atiyah Conjecture says that all L^2-Betti numbers of a finite CW-complex with fundamental group G are integers, if G is torsion free. If a stronger version of the Atiyah Conjecture holds for a group G over the group ring KG, it implies the Zero Divisor Conjecture for G, if G is torsion free [28, Remark 1.4].
Schick and Linnell [28] define a class $\mathcal{F}$ of groups, which has the characteristic property that the Atiyah Conjecture is inherited by extensions with finite or elementary amenable quotient. This implies that for groups in $\mathcal{F}$ the Zero Divisor Conjecture is inherited by extensions with finite or elementary amenable quotient.
Roughly speaking $\mathcal{F}$ is the class of groups G, defined by the following properties:

1. G has got a finite classifying space.

2. G is cohomologically complete (or p-good for all primes p).

3. all quotients G/G_n of the descending central series are torsion free.

   [in general: G has *enough nilpotent torsion free quotients*, which is explained in detail in Section 5.1].

A discrete group G is cohomologically complete, if for all primes p the canonical homomorphism

$$H^*(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism. Here $\hat{G}^p$ is the pro-p completion of G.
One of the main results of this thesis is that primitive link groups lie in $\mathcal{F}$. Schick and Linnell [28] have shown that primitive link groups have a classifying space of dimension 2, and using results of Labute [25] one can show that all quotients of the

descending central series of primitive link groups are torsion free. For further details on how this can be done refer to Theorem 5.4 on page 36 and Theorem 5.6 on page 38. So what remains to be established, to show that primitive link groups lie in $\mathcal{F}$, is the cohomological completeness. In a recent preprint Hillman, Matei and Morishita claim that link groups are cohomologically complete [17]. Using the terminology of Serre [40], they call this notion p-good. They pursue an interesting approach, using the theory of spectral sequences, in particular the Lyndon/Hochschild-Serre spectral sequence. It seemed the problem had been solved, however we had a closer look at their proof, in order to maybe transfer their methods to other cases. Unfortunately we found a gap and a counter example to their proof. Although we have contacted them several times, they do not reply to our inquiry. We have the impression that it is not easy to fix the gap, however with the additional requirement of considering abelian groups only, the proof can be fixed. This is not very interesting for link groups, as probably not many of them are abelian. There are however some other cases, which might be of interest. Discussing these, we give an extension of the repaired proof in the appendix.

The question whether primitive link groups are cohomologically complete, was again open. Based on a preprint of Linnell and Schick [29] and a recently published paper of Labute on mild pro-p groups [26], we were able to prove that primitive link groups are indeed cohomologically complete, so they belong to $\mathcal{F}$.

The notion mild (refer to Definition 3.4 on page 25) is an important tool in the proof. However mild groups are in general of interest, when considering the class $\mathcal{F}$. This interesting connection is another major result of this thesis. We have shown that mild groups have enough nilpotent torsion free quotients (Theorem 5.4 on page 36) and with respect to cohomological completeness, they share a lot of good properties as we will see in the sequel. Mildness is a natural property of a group, and there are different ways to establish, whether a group is mild or not. Applying results of Anick [2], we show that groups of Koch type (refer to Definition 5.8 on page 39) are mild (Theorem 5.9 on page 39).

As initially stated, the strong Atiyah Conjecture for a torsion free group G over KG implies the Zero Divisor Conjecture. The Zero Divisor Conjecture, which goes back to Kaplansky and is about 60 years old, says that if G is a torsion free group and K is a field, then the group ring KG does not contain zero divisors.

It has been solved for a number of cases, however at the moment it does not seem to be possible to approach it in general. So one task, when trying to solve the Zero Divisor Conjecture, is to find appropriate properties of a bunch of torsion free groups, in order

to try to solve the Zero Divisor Conjecture for all torsion free groups sharing these properties.
The conjecture is subject to different fields in mathematics such as algebra and topology, and solutions for a specific class of groups have been established from different point of views. After Passman has written his book on the structure of group rings in 1977 [35], there seems to be no overview over all cases for which the conjecture is solved. However as there are different viewpoints from which you can look at the conjecture, the results are spread widely through literature. There are a lot of properties, that imply the non-existence of zero divisors in the group ring, and so there are examples of groups in literature, for which the conjecture is actually known to be true, although it has never been stated explicitly. To give an overview over the contributions in research to the conjecture, we will give a list of the groups, for which it has been solved. We contribute a new class of groups to the list.

We now briefly explain the structure of the thesis.
In the second chapter we give some basic preliminaries, that will be needed in the sequel. In the third chapter we introduce the specific theory of Lie algebras associated to the descending central series of a group, that was used a lot in the various papers of Labute. We also introduce some of the basic tools Labute used to work with, as we will need them later on. In particular link groups and mild groups are defined. The fourth chapter is dedicated to the Zero Divisor Conjecture, and basically gives a list of all the groups for which the conjecture holds, contributing an own result to the list. The fifth chapter is dealing with the property of a group to have enough nilpotent torsion free quotients, one of the properties a group must have to lie in $\mathcal{F}$. The main result of this chapter is to show that mild groups have enough nilpotent torsion free quotients. We then show that primitive link groups and groups of Koch type are mild, so we give a large class of new examples of mild groups. Similarly the sixth chapter is dealing with cohomological completeness. When considering the class $\mathcal{F}$ of groups, it turned out, that cohomological completeness is often the crucial point. We have shown that primitive link groups are cohomologically complete. Moreover we point out that mild groups share useful properties in this context. Finally in Chapter 7 the class $\mathcal{F}$ of groups is the main subject. Results from Chapter 5 and Chapter 6 are put together to deduce that primitive link groups lie in $\mathcal{F}$. In the second section we show that for the Atiyah Conjecture to be true for all extensions with finite or elementary amenable quotient of a group in $\mathcal{F}$, we actually do not need to verify the conjecture for the group itself, but it is sufficient that the conjecture holds for specific subgroups.

1.1 The Results

We will list the new results, that have been obtained in this thesis. More details can be found in the following chapters.

THEOREM 4.13. *Let G be a torsion free group with a normal series*

$$G = G_0 \rhd G_1 \rhd G_2 \dots \text{ with } \bigcap_{i \in \mathbb{N}} G_i = \{e\}$$

such that each subquotient G_i/G_{i+1} is torsion free abelian. Then the group ring KG has no zero divisors.

This result can be found on page 30. It is an extension of a result of Bovdi (Theorem 4.3 on page 28) and gives new examples of groups that satisfy the Zero Divisor Conjecture.

Coming to the properties of the class $\mathcal{F}$ of groups, we obtained:

THEOREM 5.4. *Let $G = F/R$ be a discrete mild group. Then G has enough nilpotent torsion free quotients.*

This theorem can be found on page 36. It gives a sufficient condition for groups to have enough nilpotent torsion free quotients, one of the necessary properties for a group to lie in $\mathcal{F}$. It implies the following result, which can be found on page 38:

THEOREM 5.6. *Primitive link groups have enough nilpotent torsion free quotients.*

Taking the former into consideration, mild groups are an interesting object to study in order to find more groups that lie in $\mathcal{F}$. Using results of Koch [20] we found a new class of groups that are mild. The result can be found on page 39.

THEOREM 5.9. *Discrete groups of Koch type are mild.*

Theorem 5.4 and Theorem 5.9 imply immediately:

COROLLARY 5.14. *Groups of Koch type have enough nilpotent torsion free quotients.*

From *enough nilpotent torsion free quotients*, we come to *cohomological completeness* now. The following lemma is only a small technical result. However it is an important tool in the proof of the consecutive theorem.

LEMMA 6.8. *Let G be a discrete group. If $cd(G) = k$ and the homomorphism*

$$H^n(\hat{G}^p, M) \to H^n(G, M)$$

is surjective for $n \leq k$ for all finite discrete $\hat{G}^p$-modules M, then G is cohomologically complete.

THEOREM 6.9. *Primitive link groups are cohomologically complete.*

This result can be found on page 49. It is a particularly nice result, we were aiming for.
The following theorem is another step towards the aim to find more groups that are cohomologically complete. It also stresses the relevance of mildness with respect to the class $\mathcal{F}$ of groups.

THEOREM 6.14. *Let G be a discrete group. If G is mild and residually a finite p-group, then $cd(G) = 2$.*

Putting together Theorem 5.6 and Theorem 6.9 we deduce:

THEOREM 7.3. *Primitive link groups lie in $\mathcal{F}$.*

Our last result is a slight technical improvement of the main theorem of Linnell and Schick [28]. It says that for the Atiyah Conjecture to be true for all extensions of finite or elementary amenable quotient of a group in $\mathcal{F}$ it is sufficient to verify the conjecture for specific subgroups, instead of for the group itself.

THEOREM 7.5. *Let H be a discrete group which lies in $\mathcal{F}$, and let G_S be the inverse image of a Sylow-p subgroup G_S/H of G/H in G. Assume that KU fulfills the strong Atiyah Conjecture, where U is a subgroup of H such that*

- *U is normal in G_S*

- *G_S/U is elementary amenable*
- *H/U is torsion free*
- *lcm(G_S/U) | lcm(G_S)*

Then the Atiyah Conjecture holds for every finite extension G of H:

$$1 \to H \to G \to G/H \to 1$$

Chapter 2

Preliminaries

In this chapter we will give some basic notions, which are important in the context of this thesis. In the first section we explain the Atiyah Conjecture, because it is inherited by extensions of finite or elementary amenable quotient, if a group lies in $\mathcal{F}$, as mentioned in the Introduction. We point out however, that the Atiyah Conjecture will not play a decisive role in this thesis, as the subject of this thesis is to concentrate on the class $\mathcal{F}$ and the requirements for a group to lie in $\mathcal{F}$. The Atiyah Conjecture is hence mainly important for the general setting and gives a motivation for the class $\mathcal{F}$.
In the second section we give a brief introduction to profinite groups. The theory of profinite groups and Galois cohomology are important to understand the notion of cohomological completeness, which is one of the properties a group must have to lie in $\mathcal{F}$.

2.1 The Atiyah Conjecture

As in our context we are only interested in torsion free groups, we state the Atiyah Conjecture for torsion free groups only. First we state the conjecture [3] from a topological point of view. It says the following:

> For a discrete torsion free group G all L^2-Betti numbers of each finite CW-complex with fundamental group G are integers.

The precise definition of L^2-Betti numbers is quite technical. We refer to Lück [31, Definition 1.30 and Definition 1.16]. Note that, if G is finite and X is a free G-CW complex of finite type, then the relation of the p-th L^2-Betti number $b_p^{(2)}$ of X and the classical Betti number $b_p = \dim_{\mathbb{C}}(H_p(X;\mathbb{C}))$ of X is the following [31, Example 1.32]:

$$\frac{1}{|G|} b_p = b_p^{(2)}$$

So L^2-Betti numbers are invariants of a CW-complex. The invariants are in some sense stronger than the classical Betti numbers.

Strong Atiyah Conjecture:
A discrete torsion free group G fulfills the strong Atiyah Conjecture over KG, where K is some subring of $\mathbb{C}$, closed under complex conjugation, if:

$$\dim_G(\ker(A)) \in \mathbb{Z} \quad \textit{for all } A \in M_n(KG).$$

We call it the *strong* Atiyah Conjecture, to distinguish it from the original conjecture (refer to Atiyah [3]), which in the case $K = \mathbb{Q}$ is equivalent to $\dim_G(\ker(A)) \in \mathbb{Q}$. This is a weaker requirement. Lück has shown that for $K = \mathbb{Q}$ the strong Atiyah Conjecture is equivalent to requiring [30, Lemma 2.2]:

> For every finite CW-complex with fundamental group G, where G is torsion free, the L^2-Betti numbers fulfill $b_p^{(2)} \in \mathbb{Z}$.

In order to define $\dim_G(\ker(A))$ we regard $A \in M_n(KG)$ as a bounded operator

$$A : l^2(G)^n \to l^2(G)^n.$$

As usual we write $l^2(G) = \{\sum_{g\in G} \lambda_g g \mid \sum_{g\in G} |\lambda_g|^2 < \infty\}$
$A : l^2(G)^n \to l^2(G)^n$ is a bounded operator. The operation is induced by

$$\sum_{g\in G} \lambda_g g \cdot \sum_{h\in G} \mu_h h = \sum_{u\in G} \sum_{gh=u} \lambda_g \mu_h u.$$

Now $\ker(A)$ is a subspace of $l^2(G)^n$. More precisely we have $l^2(G)^n = \ker(A) \perp W$ for some subspace W of $l^2(G)^n$.
If $\pi_{\ker(A)}$ is the orthogonal projection onto $\ker(A)$ and $e_i \in l^2(G)^n$ is given by $e_i = (\delta_{ij})_{j=1\ldots n}$, denote the image of e_i under $\pi_{\ker(A)}$ with h_i, i. e. $h_i = (\sum_{g\in G} \alpha_{g_{ij}} g)_{j=1,\ldots,n}$, then

$$\dim_G(\ker(A)) = \sum_{i=1}^{n} \langle \pi_{\ker(A)}(e_i), e_i \rangle_{l^2(G)^n} = \sum_{i=1}^{n} \sum_{g\in G} h_i \cdot e_i = \sum_{i=1}^{n} \alpha_{1_{ii}} \in K$$

It will be important in the sequel, that the Atiyah Conjecture for torsion free groups implies the Zero Divisor Conjecture [30, Lemma 2.4].

2.2 Profinite Groups and Galois Cohomology

The theory of profinite groups is an important tool in the sequel, as we will need the notion of cohomological completeness, which is to be defined later in this chapter. For more details on these notions refer to Wilson [42]. Let us prepare the definition of a profinite group now.
The *inverse limit* of an inverse system (G_i, φ_{ij}) $i, j \in I$ of topological groups is a topological group G together with a family of compatible homomorphisms $(\varphi_i : G \to G_i)$ such that (G, φ_i) satisfies the following universal property:
Suppose $(\psi_i : H \to G_i)$ is a family of compatible homomorphisms, where H is a topological group, then there exists a unique homomorphism $\psi : H \to G$ satisfying $\varphi_i \psi = \psi_i \quad \forall i \in I$. A family of homomorphisms $(\varphi_i : G \to G_i)$ is compatible here, if $\varphi_{ij} \varphi_j = \varphi_i \; \forall i, j \in I$.
A *profinite group* is the inverse limit of an inverse system of discrete finite groups. For this definition it is important to point out that the inverse limit exists and is unique [42]. If G is the inverse limit of $\{G_i \mid i \in I\}$ we write

$$G = \varprojlim_{i \in I} G_i$$

The term *profinite* was probably introduced by Serre in the late 1950s. It stands for *pro*jective limit of *finite* groups. It might be interesting to know that profinite groups are exactly Galois groups [42, Proposition 3.1.1, p.47 and Theorem 3.3.2, p.51]. However this equivalence will not play any role in the sequel whereas the following properties are of interest, because they describe the structure of profinite groups (see for instance Wilson [42, Corollary 1.2.4, p.19]):

THEOREM 2.1. *Let G be a topological group. Then the following are equivalent:*

a) *G is profinite.*

b) *G is isomorphic to a closed subgroup of the Cartesian product of discrete finite groups.*

c) *G is compact and $\bigcap\{N \mid N \lhd_\circ G\} = \{1\}$.*

d) *G is compact and totally disconnected.*

Here $N \lhd_\circ G$ means N is open and normal in G. Using compactness it follows that every open subgroup of a profinite group has finite index.
The inverse limit of finite p-groups is also called a *pro-p group*.

2.2.1 Completions of Groups

We will need the notion of the pro-p completion of a group to define cohomological completeness. However we will start with the general concept of a completion.
Let G be any abstract group and I a non-empty filter base of normal subgroups of finite index of G. We consider the following topology on G: A subset of G is open, if it is a union of cosets gK of subgroups $K \in I$. Equipped with this topology G becomes a topological group [42, p.9]. Now the *completion* of G with respect to I consists of a profinite group $\hat{G}$ and a continuous homomorphism $j : G \to \hat{G}$ such that, whenever $\theta : G \to H$ is a continuous homomorphism to a profinite group H, there exists a unique continuous homomorphism $\hat{\theta} : \hat{G} \to H$ such that $\theta = \hat{\theta} j$. The completion of a group G with respect to I is given by

$$\hat{G} = \varprojlim_{K \in I} G/K$$

together with the map $j : g \mapsto (gK)_{K \in I}$ as one can check easily. The word *completion* makes sense, since $j(G)$ is dense in $\hat{G}$ [42]. Note that in general j is not injective.
The *profinite completion* of a group G is the completion of G with respect to the family of normal subgroups of finite index. We come to the definiton of the *pro-p completion* of a group G:

DEFINITION 2.2. *Let p be a prime. Then the pro-p completion of a group G is the completion with respect to the family of normal subgroups having p-power index. It is denoted by $\hat{G}^p$.*

It follows immediately that the pro-p completion of an arbitrary group G is a pro-p group.
Note that for distinct primes p, q the pro-p completion of $\mathbb{Z}/q\mathbb{Z}$ is the trivial group, since the only subgroup of $\mathbb{Z}/q\mathbb{Z}$ having p-power index is $\mathbb{Z}/q\mathbb{Z}$ itself. In particular j is not injective.

2.2.2 Galois Cohomology

To introduce Galois cohomology we will follow the approach via resolutions. For a different approach via chain complexes consult Wilson [42].
Let R be a ring and N an R-module. A *resolution* of N is a non-negative R-complex C with an epimorphism $\varepsilon : C_0 \to N$ such that the following sequence is exact:

$$\ldots \longrightarrow C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

If every C_i is a projective R-module, the resolution is said to be projective.
Let M be an R-module. To each resolution $C \xrightarrow{\varepsilon} N$ of N we associate a complex of abelian groups:

$$\ldots \longleftarrow \mathrm{Hom}_R(C_2, M) \xleftarrow{\delta_2'} \mathrm{Hom}_R(C_1, M) \xleftarrow{\delta_1'} \mathrm{Hom}_R(C_0, M)$$

with $\delta_i' : \varphi \mapsto \varphi\delta_i$.
Suppose now $R = \mathbb{Z}G$ is the group ring of a profinite group G. Then R is a topological ring where the topology of G is the subspace topology [42, Ex. 9.9.14]. Then the *n-th cohomology group* of $\mathbb{Z}G$ with coefficients in M is defined by:

$$\mathrm{H}^n(\mathbb{Z}G, M) = \mathrm{H}_n(\mathrm{Hom}_{\mathbb{Z}G}(P, M))$$

with $P \xrightarrow{\varepsilon} \mathbb{Z}$ a projective resolution. Note that this definition is well-defined: If $P' \xrightarrow{\varepsilon'} \mathbb{Z}$ is another projective resolution, then the R-complexes are homotopy equivalent and hence the homology groups are isomorphic [42, Proposition 9.8.3, Lemma 9.8.4, Lemma 9.8.1].
It follows the definition of the *n-th cohomology group* of a profinite group G:

DEFINITION 2.3. *Let G be a profinite group. Then* $H^n(G, M) = H_n(Hom_{\mathbb{Z}G}(P, M))$ *with* $P \xrightarrow{\varepsilon} \mathbb{Z}$ *a projective resolution.*

2.2.3 Cohomological Completeness

We have all necessary tools now to define cohomological completeness.

DEFINITION 2.4. *A discrete group G is said to be cohomologically complete, if for all primes p the canonical homomorphism:*

$$H^*(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z}) \tag{2.1}$$

is an isomorphism. $\hat{G}^p$ *is the pro-p completion of G.*

To show that there exists a canonical homomorphism $H^*(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z})$, we will investigate the cohomology groups more in detail. First of all we will show how the canonical homomorphism

$$H_n(\mathrm{Hom}_{\mathbb{Z}\hat{G}^p}(D, \mathbb{Z}/p\mathbb{Z})) \to H_n(\mathrm{Hom}_{\mathbb{Z}G}(C, \mathbb{Z}/p\mathbb{Z}))$$

is defined, where $D \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}\hat{G}^p$-modules and $C \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}G$-modules. It is immediately obvious that every $\mathbb{Z}\hat{G}^p$-module M is also a $\mathbb{Z}G$-module by defining the module map $\mathbb{Z}G \times M \to M$ as $gm \mapsto j(g)m$.
The Comparison Theorem states the following (e.g. Weibel [41, Comparison Theorem 2.2.6]):

Comparison Theorem 2.5. *Let $C \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' : M \to N$ a map in an abelian category. Then for every resolution $D \xrightarrow{\eta} N$ of N there is a chain map $f : C \to D$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. The chain map f is unique up to chain homotopy equivalence.*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'} & & \\
\cdots & \longrightarrow & D_2 & \longrightarrow & D_1 & \longrightarrow & D_0 & \xrightarrow{\eta} & N & \longrightarrow & 0
\end{array}$$

In our case we consider the abelian category of $\mathbb{Z}G$-modules, and $C \xrightarrow{\epsilon} \mathbb{Z}$ is a projective resolution of $\mathbb{Z}G$-modules and $D \xrightarrow{\eta} \mathbb{Z}$ is a projective resolution of $\mathbb{Z}\hat{G}^p$-modules.
Keep in mind that every $\mathbb{Z}\hat{G}^p$-module is also a $\mathbb{Z}G$-module and, as we will see below, every $\mathbb{Z}\hat{G}^p$-module homomorphism is also a $\mathbb{Z}G$-module homomorphism.
Let $f' : \mathbb{Z} \to \mathbb{Z}$ be the identity map. According to the above theorem there exists a chain map $f : C \to D$. So we have the following double complex:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_2 & \xrightarrow{c_2} & C_1 & \xrightarrow{c_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'} & & \\
\cdots & \longrightarrow & D_2 & \xrightarrow{d_2} & D_1 & \xrightarrow{d_1} & D_0 & \xrightarrow{\eta} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

To this chain complex we can associate the following chain complex:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathbb{Z}G}(C_0, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{c_1'} & \mathrm{Hom}_{\mathbb{Z}G}(C_1, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{c_2'} & \mathrm{Hom}_{\mathbb{Z}G}(C_2, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{c_3'} & \cdots \\
\uparrow{\scriptstyle F_0} & & \uparrow{\scriptstyle F_1} & & \uparrow{\scriptstyle F_2} & & \\
\mathrm{Hom}_{\mathbb{Z}\hat{G}^p}(D_0, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{d_1'} & \mathrm{Hom}_{\mathbb{Z}\hat{G}^p}(D_1, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{d_2'} & \mathrm{Hom}_{\mathbb{Z}\hat{G}^p}(D_2, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{d_3'} & \cdots
\end{array}$$

with the canonical definition of F_i:

$$\begin{array}{rcl} F_i : \mathrm{Hom}_{\mathbb{Z}\hat{G}^p}(D_i, \mathbb{Z}/p\mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}G}(C_i, \mathbb{Z}/p\mathbb{Z}) \\ (\varphi : D_i \to \mathbb{Z}/p\mathbb{Z}) & \mapsto & \varphi \circ f_i \end{array}$$

Note that F is a chain map, since f is a chain map. To verify that this is well-defined we will show that every $\mathbb{Z}\hat{G}^p$-module homomorphism is also a $\mathbb{Z}G$-module homomorphism.
Let $\varphi : D_i \longrightarrow \mathbb{Z}/p\mathbb{Z}$ be a $\mathbb{Z}\hat{G}^p$-module homomorphism, i. e. $g'\varphi(a) = \varphi(g'a)\ \forall g' \in \mathbb{Z}\hat{G}^p, a \in D_i$. For all $g \in \mathbb{Z}G, a \in D_i$ we have

$$\begin{array}{rcl} g\varphi(a) & = & j(g)\varphi(a) \\ & = & \varphi(j(g)a) \\ & = & \varphi(ga) \end{array}$$

and so φ is a $\mathbb{Z}G$-module homomorphism. (The converse is not true.)
So D_i as a $\mathbb{Z}\hat{G}^p$-module is also a $\mathbb{Z}G$-module, and φ as a $\mathbb{Z}\hat{G}^p$-module homomorphism is also a $\mathbb{Z}G$-module homomorphism. And hence $\varphi \circ f_i$ is also a $\mathbb{Z}G$-module homomorphism.
Now the homomorphism we are looking for, can be defined in the following way:

$$\begin{array}{rcl} H^i(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) = \ker(d'_{i+1})/\mathrm{im}(d'_i) & \to & \ker(c'_{i+1})/\mathrm{im}(c'_i) = H^i(G, \mathbb{Z}/p\mathbb{Z}) \\ \varphi\mathrm{im}(d'_i) & \mapsto & F_i(\varphi)\mathrm{im}(c'_i) \end{array}$$

This map is well-defined:
Suppose $\varphi_1\mathrm{im}(d'_i) = \varphi_2\mathrm{im}(d'_i)$. Then there exists an element $\psi \in \mathrm{im}(d'_i)$ such that $\varphi_2\psi = \varphi_1$. Now

$$\begin{array}{rcl} \varphi_1\mathrm{im}(d'_i) \mapsto F_i(\varphi_1)\mathrm{im}(c'_i) & = & F_i(\varphi_2\psi)\mathrm{im}(c'_i) \\ & = & F_i(\varphi_2)F_i(\psi)\mathrm{im}(c'_i) \\ & = & F_i(\varphi_2)\mathrm{im}(c'_i) \end{array}$$

where the last equality holds, since $\psi \in \mathrm{im}(di')$, and therefore there exists an element $\bar{\psi} \in \mathrm{Hom}_{\mathbb{Z}\hat{G}^p}(D_{i-1}, \mathbb{Z}/p\mathbb{Z})$ such that $d'_i(\bar{\psi}) = \psi$. Consequently $F_i(\psi) = F_i(d'_i(\bar{\psi})) = c'_i(F_{i-1}(\bar{\psi}))$, and thus $F_i(\psi) \in \mathrm{im}(c'_i)$.
It is straight forward to show that it is a homomorphism.

Chapter 3

Lie Algebras, Link Groups and Mild Groups

We will come to more advanced specific theory now that will be needed in the course of the thesis. In the first section of this chapter we introduce the notion of the Lie algebra associated to the descending central series of a group. The investigation of this Lie algebra will become a useful tool, when we consider the class $\mathcal{F}$ of groups. As we will see in Chapter 5, there exists a sufficient condition for groups to have enough nilpotent torsion free quotients that involves this Lie algebra. Moreover this notion will be necessary to define a mild group. The purpose of the second section is to understand the notion of a primitive link group. Finally the third section briefly introduces mild groups. Primitive link groups and certain mild groups are important examples of groups, that we will show lie in $\mathcal{F}$.

3.1 Lie Algebras

In this context Lie algebras are considered as abstract algebraic structures as in Humphreys [19] and Bourbaki [7]. In particular a Lie algebra is not necessarily a vector space:

DEFINITION 3.1. *A module L over a commutative ring R with bracket operation $L \times L \to L$, $(x, y) \mapsto [xy]$ is a Lie algebra, if the following hold:*

1. *The bracket operation is bilinear;*

2. $[xx] = 0$ *for all $x \in L$;*

3. $[x[yz]] + [y[zx]] + [z[xy]] = 0$ *for all $x, y, z \in L$ ("Jacobi identity").*

Let R be a commutative ring. A family $(x_i)_{i\in I}$ of elements of an R-module M is free or linearly independent, if $\sum_{i\in J} a_i x_i = 0$ with $a_i \in R$ implies $a_i = 0 \ \ \forall i \in J$, where J is any finite subset of I. M is called a *free module*, if it contains a free system of generators. The maximal number of free elements in M is called the *rank* of M. The rank of a Lie algebra will be needed to introduce the notion of the Poincaré series associated to a graded Lie algebra.

The *universal enveloping algebra* is important in representation theory and plays a decisive role in our context as well (for instance in the definition of a mild group). Let L be a Lie algebra over R. A universal enveloping algebra of L is an associative algebra $\mathcal{U}(L)$ with 1 over R with a homomorphism $i : L \to \mathcal{U}(L)$ such that

$$i([xy]) = i(x)i(y) - i(y)i(x) \ \ \forall x, y \in L. \tag{3.1}$$

If $\mathcal{A}$ is an associative algebra with 1 and $j : L \to \mathcal{A}$ a homomorphism satisfying (3.1), then there exists a unique algebra homomorphism $\phi : \mathcal{U}(L) \to \mathcal{A}$ (sending 1 to 1) such that $\phi \circ i = j$. The universal enveloping algebra exists and is unique. It is given by

$$\mathcal{U}(L) = \bigoplus_{i=0}^{\infty} L^{\otimes i}/J \tag{3.2}$$

where J is the ideal generated by $\{x \otimes y - y \otimes x - [xy] \mid \forall x, y \in L\}$. Note that the canonical mapping $L \to \mathcal{U}(L)$ is injective, hence L can be considered as a subalgebra of $\mathcal{U}(L)$ [19, p.92, Corollary B].

3.1.1 The Lie Algebra associated to the Descending Central Series of a Group

Let G be a group. Then the *descending central series* of G is the following sequence of subgroups $\{G_n, \ n \in \mathbb{N}\}$:

$$G_1 = G, \ \ G_{n+1} = [G, G_n] \ \text{ for } n \geq 2.$$

The descending central series of a group G is a filtration of G, since it satisfies the following:

1. $G_1 = G$
2. $G_{n+1} \subseteq G_n$

3. $[G_n, G_k] \subseteq G_{n+k}$

The last inclusion is not immediately obvious. It is obtained by using commutator identities. For further details refer to Bourbaki [6].
The abelian group associated to a filtration is of the form:

$$gr(G) = \bigoplus_{n=1}^{\infty} gr_n(G) = \bigoplus_{n=1}^{\infty} G_n/G_{n+1}.$$

It has the structure of a graded Lie algebra over $\mathbb{Z}$. The bracket operation is induced by the commutator operation in G in the following way:

$$\left[\sum_{n\in\mathbb{N}} g_n G_{n+1}, \sum_{n\in\mathbb{N}} h_n G_{n+1}\right] = \sum_{n\in\mathbb{N}} \sum_{i=1}^{n} [g_i, h_{n-i}] G_{n+1}.$$

This Lie algebra is called the *Lie algebra associated to the descending central series of a group* or just the Lie algebra associated to a group. $gr_n(G)$ is called the n-th homogeneous component of the Lie algebra. The concept to construct a Lie algebra associated to a filtration of a group is universal and works for other examples, as for instance the lower p-central series as well [24].
In the following let F be the free group on m generators. Following Labute [24] we consider the following: Let

$$G = \langle x_1, \ldots, x_m \mid r_1, \ldots r_d \rangle = F/R \tag{3.3}$$

be a finitely presented group. For $x \in F, x \neq 1$, there is a largest integer $n = n_x \geq 1$ such that $x \in F_{n_x}$. This integer is called the *weight of* x with respect to $(F_n)_{n\in\mathbb{N}}$. We call the image of x in $gr_n(F)$ the *initial form of* x. Now let ρ_i be the initial form of r_i for $i = 1, \ldots, d$. Let $\mathfrak{r} = \langle \rho_1, \ldots, \rho_d \rangle$ be the ideal of $gr(F)$ that is generated by $\{\rho_1, \ldots, \rho_d\}$. Now $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a $U(gr(F)/\mathfrak{r})$-module. The module map is induced by the adjoint representation. It is hence given by:

$$\begin{aligned} U(gr(F)/\mathfrak{r}) \times \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] &\rightarrow \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] \\ ((\sum_{i=0}^{k} x_i)J, x[\mathfrak{r}, \mathfrak{r}]) &\mapsto (\sum_{i=0}^{k} [\xi_{i1}[\xi_{i2}[\ldots[\xi_{ii}, x]]\ldots])[\mathfrak{r}, \mathfrak{r}] \end{aligned}$$

where $(\sum_{i=0}^{k} x_i)J$ is an arbitrary element of $U(gr(F)/\mathfrak{r}) = \bigoplus_{i=0}^{\infty} (gr(F)/\mathfrak{r})^{\otimes i}/J$, i. e. $k \in \mathbb{N}$ is arbitrary and $x_i \in (gr(F)/\mathfrak{r})^{\otimes i}$ with $x_i = \xi_{i1}\mathfrak{r} \otimes \xi_{i2}\mathfrak{r} \otimes \ldots \otimes \xi_{ii}\mathfrak{r}$ for $i = 1, \ldots k$. Note that $x_0 \in \mathbb{Z}$ and $(x_0, x[\mathfrak{r}, \mathfrak{r}]) \mapsto x_0 x[\mathfrak{r}, \mathfrak{r}] = \underbrace{x[\mathfrak{r}, \mathfrak{r}] + \ldots + x[\mathfrak{r}, \mathfrak{r}]}_{x_0-times}$.

In general $gr(G) = gr(F/R) \neq gr(F)/\mathfrak{r}$, unless the relators $r_1, \ldots, r_d$ satisfy specific independence conditions as the following [25, Theorem 1 and remarks, p.52]:

THEOREM 3.2. *If* $gr(F)/\mathfrak{r}$ *is a free* $\mathbb{Z}$*-module and* $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ *is a free* $U(gr(F)/\mathfrak{r})$*-module on* $\rho_1[\mathfrak{r},\mathfrak{r}], \ldots, \rho_d[\mathfrak{r},\mathfrak{r}]$*, then* $gr(G) = gr(F)/\mathfrak{r}$*.*

A group satisfying these properties will be called mild later on (Definition 3.4 on the following page). Labute however used these independence properties long before the notion *mild* had been introduced, as one can see in his various papers quoted in the bibliography.

3.2 Link Groups

In the following we will only introduce the basic notions, that will be needed in the sequel. For more details see for instance Rolfsen [37].

DEFINITION 3.3. *A link L is the image of a continuous map from a number of copies of* S^1 *into* S^3*. The number of copies of* S^1*, is called the number of components of the link. We speak of an m-component link for instance.*

A link is said to be *tame*, if it has a finite number of components that are pairwise disjoint and polygonal curves, i. e. they do not have an infinite sequence of smaller knots, for instance. All links we consider are assumed to be tame.
One can also think of a link as a closed braid.
The *linking number* l_{ij} of two components i and j is roughly speaking the number of times that each component turns around the other component. More formally, for two components i and j of a link, at each point at which i crosses under j count +1 if j crosses from right to left and -1 if it crosses from left to right with respect to i. The sum of all these numbers is called the linking number (or linking index) of i and j.
The *linking diagram* of a link L is the edge-weighted graph, whose vertices are the components of L. Two vertices are joined by an edge, whose weight is the linking number of the relevant vertices.
The *link group* G_L of a link L is the fundamental group $G_L = \pi_1(S^3 \setminus L)$ of the complement of L in S^3.
A link is said to be *primitive*, if its linking diagram is connected mod p for every prime p.

3.3 Mild Groups

The notion of a mild group has first been introduced by Anick in 1987 [2]. Labute has recently studied mild pro-p groups [26]. However without labelling them as mild groups, Labute had studied them in his earlier papers [22, 23, 24, 25].

DEFINITION 3.4. *A discrete group G is said to be mild, if there exists a presentation* $G = F/R = \langle x_1, \ldots, x_m \mid r_1, \ldots r_d \rangle$ *such that the set of relations is mild, i. e.*

1. *the quotient Lie algebra* $gr(F)/\mathfrak{r}$ *is a free* $\mathbb{Z}$*-module;*

2. $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ *is a free* $U(gr(F)/\mathfrak{r})$*-module with basis* $\{\rho_1[\mathfrak{r}, \mathfrak{r}], \ldots, \rho_d[\mathfrak{r}, \mathfrak{r}]\}$*;*

If the above holds, the sequence $\rho_1, \ldots, \rho_d$ *is said to be strongly free [26, Definition 3.1].*

Note that the definition is stated with respect to the descending central series, i. e. $gr(F)$ is the $\mathbb{Z}$-Lie algebra associated to the descending central series of F. It is a local condition and makes also sense, when considering other filtrations of the group G. The conditions may have to be adjusted, as for instance, when considering the lower p-central series, the resulting Lie algebra is a $\mathbb{Z}/p\mathbb{Z}[x]$-Lie algebra [24, Section 5].
We will state the definition of a mild pro-p group G, as we will need it in the sequel:

DEFINITION 3.5. *A pro-p group G is said to be mild, if there exists a presentation* $G = F/R = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_d \rangle$ *such that the set of relations is mild, i. e. :*

1. *the quotient Lie algebra* $gr(F)/\mathfrak{R}$ *is a free* $\mathbb{Z}_p$*-module;*

2. $\mathfrak{R}/[\mathfrak{R}, \mathfrak{R}]$ *is a free* $U(gr(F)/\mathfrak{R})$*-module with basis* $\{\rho_1[\mathfrak{R}, \mathfrak{R}], \ldots, \rho_d[\mathfrak{R}, \mathfrak{R}]\}$.

Here $\mathfrak{R}$ denotes the ideal of $gr(F)$, that is generated by the initial forms of the r_i. We write $\mathfrak{R}$ instead of $\mathfrak{r}$ to distinguish it from the discrete setting. In the pro-p setting all subgroups are considered to be closed. Labute has stated the definition of a mild pro-p group with respect to the lower p-central series [26, Definition 1.1].

Chapter 4

The Zero Divisor Conjecture

Throughout this chapter G is a torsion free group and K is a field unless otherwise stated.
First recall again the Zero Divisor Conjecture:

Zero Divisor Conjecture [Kaplansky 1940s]: *Suppose G is a torsion free group and K is a field. Then the group ring KG has no zero divisors.*

The conjecture is also interesting and possibly true, if K is any integral domain.
Note that if G has torsion elements one can easily construct zero divisors:
Suppose $g \in G$ has order n. Then $(1 - g)(1 + g + g^2 + \ldots + g^{n-1}) = 0$.
For trivial reasons the Zero Divisor Conjecture is inherited by subgroups.

4.1 History and Results

As we have mentioned in the Introduction, there does not seem to be a list of all groups, for which the Zero Divisor Conjecture holds, that is up to date. For this reason, in the sequel we will try to list all groups for which the conjecture holds. Note that some of the results became redundant, because later a stronger result was proven. We will start off with Higman [16], who defined the following notion:

DEFINITION 4.1. *A group G is said to be indexed if there exists a non-trivial homomorphism $G \to \mathbb{Z}$. Furthermore a group G is said to be locally indicable (or indicable throughout) if every non-trivial subgroup of G can be indexed.*

The following theorem is due to Higman [16, Theorem 12] and was proved in 1940:

THEOREM 4.2. *If the group G is locally indicable, the group ring KG has no zero divisors.*

Probably Higman observed the non-existence of zero divisors in these group rings before the conjecture in general had been stated.

With a subnormal series of a group G we mean a series

$$G = G_0 \triangleright G_1 \triangleright G_2 \dots$$

such that G_{i+1} is a normal subgroup of G_i for all $i \in \mathbb{N}$. Note that the G_i are not necessarily normal in G.
A normal series of a group G will be a series

$$G = G_0 \triangleright G_1 \triangleright G_2 \dots$$

such that G_i is normal in G for all $i \in \mathbb{N}$.
[Note that sometimes a subnormal series is also called a normal series and a normal series is also called an invariant series.]
If $\star$ is any property of a group. Then a group G is said to be residually $\star$, if G has a normal series

$$G = G_0 \triangleright G_1 \triangleright G_2 \dots$$

such that $\bigcap_{i \in \mathbb{N}} G_i = \{e\}$ and all quotients G/G_i have the property $\star$.
If G is finitely generated this is equivalent to requiring: For every $g \in G$ there exists a normal subgroup H such that $gH \neq H$ and G/H is a $\star$-group.
In [34, Theorem 10.3] Passman states the following theorem, which is due to Bovdi and was proved in 1960; [the original reference is in russian, so we referred to Passman instead]:

THEOREM 4.3. *Let G be a torsion free group with a finite subnormal series*

$$G = G_0 \triangleright G_1 \triangleright G_2 \dots \triangleright G_n = \{e\}$$

such that each subquotient G_i/G_{i+1} is torsion free abelian. Then the group ring KG has no zero divisors.

For example torsion free nilpotent groups have this property and so satisfy the Zero Divisor Conjecture. This becomes clear using the upper central series (refer to Passman [35, p. 587/588, proof of Lemma 1.6]).
We will need the definition of left-orderable:

DEFINITION 4.4. *A group G is called left-orderable, if there exists a strict total left-invariant ordering < on G, i.e.*

$$g < h \quad \Rightarrow \quad fg < fh \text{ for all } f, g, h \in G.$$

G is called orderable, if it admits an ordering that is both left- and right-invariant.

In 1968 LaGrange and Rhemtulla stated the following [27]:

LEMMA 4.5. *Let G be a left-orderable group. Then the group ring KG contains no zero divisors.*

For example free groups are left-orderable and so satisfy the Zero Divisor Conjecture. We have the following theorem (see for instance [8]):

THEOREM 4.6. *If G is orderable, then it is locally indicable. If G is locally indicable, then it is left-orderable. Neither of these implications is reversible.*

We will need the following definition:

DEFINITION 4.7. *A group G is said to be a positive one-relator group, if there exists a presentation of G with only one relation, that has no negative exponents.*

Baumslag proved the following Lemma in 1971 [4]:

LEMMA 4.8. *Let G be a positive one-relator group. Then the group ring KG has no zero divisors.*

DEFINITION 4.9. *A group G is said to be supersolvable (or supersoluble) if it has a finite normal series*

$$G = G_0 \rhd G_1 \rhd G_2 \ldots \rhd G_n = \{e\}$$

with cyclic subquotients G_i/G_{i+1}.

In 1973 Formanek could prove the following [15]:

LEMMA 4.10. *If G is supersolvable, then the group ring KG has no zero divisors.*

DEFINITION 4.11. *A group G is said to be a unique-product group (UP-group), if for any two nonempty finite subsets* A, B *of G there exists at least one element* $x = ab \in AB$ *such that* $x \notin (A \setminus \{a\})(B \setminus \{b\})$.

Some authors call a unique-product group an Ω-group.
In 1974 Cohen showed the following Lemma [12, Lemma 0.1]:

LEMMA 4.12. *If G is a UP-group, then the group ring KG has no zero divisors.*

For example abelian groups are UP-groups, and so satisfy the Zero Divisor Conjecture. Moreover locally indicable groups and left-orderable groups are UP-groups. It is interesting that the proof of the Lemma is elementary and very short, although the result is quite strong as we will see soon.
Using the above Lemma, we will show that a variation of Theorem 4.3 of Bovdi holds as well:

THEOREM 4.13. *Let G be a torsion free group with a normal series*

$$G = G_0 \rhd G_1 \rhd G_2 \ldots \text{ with } \bigcap_{i \in \mathbb{N}} G_i = \{e\}$$

such that each subquotient G_i/G_{i+1} is torsion free abelian. Then the group ring KG has no zero divisors.

Proof:
Suppose G has a normal series

$$G = G_0 \rhd G_1 \rhd G_2 \ldots \text{ with } \bigcap_{i \in \mathbb{N}} G_i = \{e\}$$

such that the subquotients G_i/G_{i+1} are torsion free abelian.
We will show inductively that G/G_i is a UP-group for all i.
First of all consider the following short exact sequence:

$$1 \to G_0/G_1 \to G/G_1 \to G/G_0 \to 1$$

By assumption G_0/G_1 is torsion free abelian, and hence by a proposition of Cohen [12, Proposition 1.3] it is a UP-group:

PROPOSITION 4.14. *Torsion free abelian groups are UP-groups.*

$G/G_0 = \{e\}$ is a UP-group as well and therefore by a theorem of Cohen [12, Theorem 1.4] G/G_1 is a UP-group as well:

THEOREM 4.15. *If $1 \to K \to G \to Q \to 1$ is an exact sequence, and K and Q are UP-groups, then G is a UP-group as well.*

Suppose the assumption is true for G/G_{i-1}. Hence we obtain:

$$1 \to G_{i-1}/G_i \to G/G_i \to G/G_{i-1} \to 1$$

with G_{i-1}/G_i torsion free abelian by assumption, and hence a UP-group. Moreover G/G_{i-1} is a UP-group by induction hypothesis. Hence quoting Cohen again, G/G_i is a UP-group as well.
So G has a normal series with trivial intersection and the quotients G/G_i are UP-groups for all $i \in \mathbb{N}$, i. e. G is residually UP. In the sequel we will see that this implies that G is in fact UP itself.
Let A, B be finite nonempty subsets of G. Now for arbitrary $x, y \in A \cup B$ there exists a group G_n such that $y^{-1}xG_n \neq G_n$. (This holds, since the G_i intersect in the trivial subgroup.) Consequently $xG_n \neq yG_n \in G/G_n$, where n depends only on x and y. There are only finitely many such pairs, since A and B are finite by assumption, and we denote the largest of these indices n by n_0.
Now assume there is no unique product in AB. Then for every pair $a \in A, b \in B$ there exist elements $x_{ab} \in A, y_{ab} \in B$ with $ab = x_{ab}y_{ab}$. But then we have $aG_{n_0}bG_{n_0} = abG_{n_0} = x_{ab}y_{ab}G_{n_0} = x_{ab}G_{n_0}y_{ab}G_{n_0}$. So for every pair aG_{n_0}, bG_{n_0} with $aG_{n_0}, bG_{n_0} \in G/G_{n_0}, \ \ a \in A, b \in B$ the product is not unique.
This however is a contradiction to G/G_{n_0} being a UP-group. Therefore the assumption is false and there must be a unique product in AB. So the group G, as given in Theorem 4.13 on the previous page is a UP-group and hence the group ring KG has no zero divisors. □

We will need the following definition:

DEFINITION 4.16. *A group G is said to be polycyclic-by-finite, if it has a finite subnormal series*

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{e\}$$

with G_i/G_{i+1} either infinite cyclic ($\cong \mathbb{Z}$) or finite.

In 1976 Farkas and Snider proved the following [14]:

LEMMA 4.17. *Let G be polycyclic-by-finite. Then the group ring KG has no zero divisors.*

To be more precise they proved the lemma if K is a field of characteristic 0. In 1980 the result was extended by Cliff to fields of arbitrary characteristic [10].
From a rather algebraic point of view we will come to a topological perspective now. Boyer, Rolfsen and Wiest have recently published a paper in which they investigate the fundamental group of 3-manifolds (also referred to as manifold groups) [8]. We

will apply their results to the case of link groups, since the complement of a link is a 3-manifold. Furthermore link groups are nice manifold groups in the sense that the associated manifolds are compact, connected and the boundary is known to consist of as many tori as the link has components. First of all recall the notion of Betti numbers (for more details see for instance Bredon [9]):

DEFINITION 4.18. *Let M be any topological space. If the i-th homology group $H_i(M)$ is finitely generated, then its rank is called the i-th Betti number. It is labelled $b_i(M)$.*

According to Boyer, Rolfsen and Wiest [8, Lemma 3.3] the following is a well-known result:

LEMMA 4.19. *If M is a compact 3-manifold and $\partial M \neq \emptyset$ and does not contain any S^2 or P^2 components, then $b_1(M) > 0$.*

Note that in particular this holds for link groups.
The following theorem is a result of Howie and Short [18] that is also quoted in Boyer-Rolfsen-Wiest [8, Theorem 3.1]:

THEOREM 4.20. *Let M be a compact, connected, P^2-irreducible 3-manifold, with $\pi_1(M)$ non-trivial. Then $\pi_1(M)$ is locally indicable if and only if $b_1(M) > 0$.*

LEMMA 4.21. *Let $G = \pi_1(M)$ be a non-trivial 3-manifold group, where M is compact, connected and the boundary is nonempty and does not contain any P^2 and S^2 components. Then KG has no zero divisors.*

In particular consider $G = G_L$ to be the group of an arbitrary link L.
Then $S^3 \setminus L$ is a compact 3-manifold whose nonempty boundary does not contain any S^2 or P^2-components.
So by Lemma 4.19 we have $b_1(S^3 \setminus L) > 0$.
Now Theorem 4.20 applies, and so the link group $\pi_1(S^3 \setminus L) = G_L = G$ is locally indicable. According to Higman (Theorem 4.2) locally indicable groups satisfy the Zero Divisor Conjecture and hence we can conclude:

COROLLARY 4.22. *Link groups satisfy the Zero Divisor Conjecture.*

This was probably known already, since Boyer, Rolfsen and Wiest stated that link groups are left-orderable [8, Corollary 3.5], though we did not find a reference stating it explicitly.
So putting the results together, the Zero Divisor Conjecture holds for the following list of groups (all groups are assumed to be torsion free):

- locally indicable groups
- groups having a finite subnormal series with torsion free abelian subquotients
- groups having a normal series that intersects trivially with torsion free abelian subquotients
- left-orderable groups
- positive one-relator groups
- supersolvable groups
- unique product groups
- polycyclic-by-finite groups
- ”nice” 3-manifold groups (see above)

This list includes for example abelian groups, free groups, nilpotent groups, solvable groups and link groups.

4.2 Further Remark

The question came up, whether all torsion free groups are unique product groups. This is not known, and if so, one could reduce the zero divisor question to this question. It is known however that groups with torsion elements are not unique product groups [12, Remark after Lemma 0.1]. If G is a group that has torsion elements, then take $A = B$ to be a nontrivial finite subgroup of G. Then AB does not have a unique product.
Furthermore Cohen [12] has shown that for finite subsets A, B of a torsion free group G, AB contains a unique product, if $|AB| \leq 24$. So we know, if a torsion free group G is not a unique product group, then the finite subsets A, B of G that do not contain a unique product, must satisfy $|AB| > 24$. His proof is very straight forward and elementary, using a few technical results only. Assuming AB has no unique product, he chooses $(x, y) \in A \times B$ and lists all products in $A \setminus \{x\} \times B \setminus \{y\}$ including as well the contradiction given if xy equals that product. (Such as elements that were assumed to be distinct turn out to be equal). He does so for $|A| = 3, \ |B| = 3, 4, 5, 6, 7$ and $|A| = 4, \ |B| = 4, 5, 6$, whereas he does not include the proofs for two of these cases,

because they are ”too long and too messy”. For these cases he refers to earlier unpublished results of his own [11].

Another interesting connection of the existence of zero divisors is the question whether there are non-trivial units in the group ring.

DEFINITION 4.23. *A unit $u \in KG$ is said to be trivial, if $u = kg$ for some $k \in K$, $k \neq 0$ and $g \in G$.*

It is not known whether the group ring KG of a torsion free group G and a field K can have non-trivial units. However if the group ring of a torsion free group contains zero divisors, then it also contains non-trivial units [35, Chapter 13, Lemma 1.2]. And if KG has only trivial units and G is torsion free, then KG has no zero divisors [35, Chapter 13, Lemma 1.2 and Lemma 1.9]. However we do not know about the converses.
Furthermore we will state the following remarkable theorem which is well-known in this field. See for instance Passman [35, Lemma 1.2].

THEOREM 4.24. *Suppose G is a torsion free group. Then the group ring KG has zero divisors if and only if it has an element of square zero.*

So the search for zero divisors can be reduced to the search for elements of square zero. However at the moment this technical result does not seem to help in approaching the Zero Divisor Conjecture.

Chapter 5

Enough Nilpotent Torsion Free Quotients

One property for a group G to lie in $\mathcal{F}$ is that it has enough nilpotent torsion free quotients. In the Introduction we only mentioned a sufficient condition for G to have enough nilpotent torsion free quotients. We will give the precise definition in the first section of this chapter. In the second section we prove that mild groups have enough nilpotent torsion free quotients, and in the last section we will give examples of groups that are mild and hence, applying our previous result, have enough nilpotent torsion free quotients. It is shown that primitive link groups and groups of Koch type are mild.

5.1 Definition

To state the definition of a group having enough nilpotent torsion free quotients (see Linnell-Schick [28, Theorem 4.52]), we will need the following:
Let G be a group. The *descending central series* of G is the following sequence of subgroups $\{G_n,\ n \in \mathbb{N}\}$:

$$G_1 = G, \quad G_{n+1} = [G, G_n] \text{ for } n \geq 2.$$

DEFINITION 5.1. *A subgroup U of G is said to be a characteristic subgroup, if it is invariant under all automorphisms of G.*

The class of elementary amenable groups is the smallest class of groups containing all finite and cyclic groups, that is closed under extensions and directed unions. Using the notation of Linnell-Schick ([28]) for a group G we define

$$\mathcal{A}_G = \{U \mid U \lhd G,\ G/U \text{ torsion free and elementary amenable}\}$$

DEFINITION 5.2. *A subset $\mathcal{A}'_G$ of $\mathcal{A}_G$ is called exhaustive, if*

1. *$U, V \in \mathcal{A}'_G$ implies $U \cap V \in \mathcal{A}'_G$.*
2. *for every $n \in \mathbb{N}$ there exists a $U_n \in \mathcal{A}'_G$ such that $U_n \subseteq G_n$*
3. *each normal subgroup $W \lhd G$ of finite p-power index contains all but finitely many of the elements of $\mathcal{A}'_G$.*

DEFINITION 5.3. *A group G is said to have enough nilpotent torsion free quotients, if $\mathcal{A}_G$ contains an exhaustive subset $\mathcal{A}'_G$ containing only characteristic subgroups U of G such that G/U is nilpotent.*

5.2 Mild Groups

The property to have enough nilpotent torsion free quotients is satisfied for instance if all quotients G/G_n of the descending central series are torsion free [28].
We will show that mild groups satisfy this condition. The notation we will use here has been introduced in Chapter 3.

THEOREM 5.4. *Let $G = F/R$ be a discrete mild group. Then G has enough nilpotent torsion free quotients.*

Proof:
First recall the notion of a mild group:

DEFINITION 5.5. *A group G is said to be mild, if there exists a presentation $G = F/R = \langle x_1, \ldots, x_m \mid r_1, \ldots r_d \rangle$ such that the set of relations is mild, i. e.*

1. *the quotient Lie algebra $gr(F)/\mathfrak{r}$ is a free $\mathbb{Z}$-module;*
2. *$\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free $U(gr(F)/\mathfrak{r})$-module with basis $\{\rho_1[\mathfrak{r}, \mathfrak{r}], \ldots, \rho_d[\mathfrak{r}, \mathfrak{r}]\}$.*

If G is mild, i. e. has the above properties, then $gr(G) = gr(F)/\mathfrak{r}$ [24, Theorem 1].
Now consider the following: By Labute we have $gr(G) = gr(F)/\mathfrak{r}$,

$\Rightarrow gr(G)$ is a free $\mathbb{Z}$-module.
$\Rightarrow gr(G)$ is torsion free.
$\Rightarrow G_i/G_{i+1}$ is torsion free for $i \in \mathbb{N}$.

Let

$$1 \to K \overset{i}{\to} G \overset{q}{\to} Q \to 1$$

be an arbitrary short exact sequence with K and Q torsion free. Then G is as well torsion free:

Suppose there exists an element $x \in G$ with $x \neq 1$ such that $x^n = 1$.

Then $q(x)^n = q(x^n) = q(1) = 1$ and therefore $x \in \ker(q)$, since $q(x)^n = 1$ and Q is torsion free.

However the sequence is exact and hence $x \in \mathrm{im}(i)$.

So there exists an element $y \in K$ with $i(y) = x$ and $y^n = 1$, since $1 = x^n = [i(y)]^n = i(y^n)$, and i is injective.

Hence $y = 1$, because K is torsion free.

It follows that $x = 1$, since i is a homomorphism.

So we have shown that G is torsion free.

We will show inductively that G/G_i is torsion free:

Start of Induction:

$$1 \to \underbrace{G_1/G_2}_{\text{torsion free}} \to G/G_2 \to \underbrace{G/G_1}_{=\{e\}} \to 1$$

so G/G_2 is torsion free.

Induction Step:

Assume that G/G_i is torsion free.

$$1 \to \underbrace{G_i/G_{i+1}}_{\substack{\text{torsion free}\\ \text{by assumption}}} \to G/G_{i+1} \to \underbrace{G/G_i}_{\substack{\text{torsion free}\\ \text{by induction hypothesis}}} \to 1$$

and so G/G_{i+1} is torsion free.

$\Rightarrow G/G_i$ is torsion free for all $i \in \mathbb{N}$, and so G has enough nilpotent torsion free quotients. $\square$

5.3 Examples

We will give some examples of groups that are mild and hence have enough nilpotent torsion free quotients.

5.3.1 Primitive Link Groups

THEOREM 5.6. *Primitive link groups have enough nilpotent torsion free quotients.*

Proof:
According to Labute [25, Theorem 2], we have the following result for a primitive link group G:

> $gr(F)/\mathfrak{r}$ is a free $\mathbb{Z}$-module and $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free $U(gr(F)/\mathfrak{r})$-module, such that $gr(G) = gr(F)/\mathfrak{r}$.

Hence primitive link groups are mild. Applying Theorem 5.4, we know that for primitive link groups the quotients G/G_n of the descending central series are torsion free, and so primitive link groups have enough nilpotent torsion free quotients. □

5.3.2 Groups of Koch Type

As we have seen in Section 5.2 on page 36, mild groups have enough nilpotent torsion free quotients. Taking this into consideration, mild groups are an interesting object to study in order to find more examples of groups that lie in $\mathcal{F}$. We will introduce groups of Koch type now and show that they are mild.
Let F be the free pro-p group on m generators. Let $A = \mathbb{Z}/p\mathbb{Z}[[x_1-1,\ldots x_m-1]]$ be the free power series ring. We define the evaluation w of Lazard of A of type $\tau_1,\ldots,\tau_m$, with $\tau_i \in \mathbb{R}^+, i = 1,\ldots,m$:

$$w(x_i - 1) = \tau_i, \quad i = 1,\ldots,m.$$

For a monom $M_{i_1\ldots i_n} = (x_{i_1} - 1)\ldots(x_{i_n} - 1)$ we have $w(M_{i_1\ldots i_n}) = \tau_{i_1} + \ldots + \tau_{i_n}$. And for an arbitrary power series $\sum_M \lambda_M M \in A$, where M ranges over all monoms,

$$w(\sum_M \lambda_M M) = \min\{w(M) \mid \lambda_M \neq 0\}.$$

where $\min \emptyset = \infty$ as usual.

EXAMPLE 5.7 (Koch's Example)**.** *Let w be the Zassenhaus-filtration of A, i.e. $w(x_i - 1) = \tau_i = 1, \quad i = 1, \dots, m$. Let*

$$r_i = (x_m x_i a_{3i} \dots a_{n_i i}) r_i' \text{ with } a_{3i}, \dots a_{n_i i} \in \{x_1, \dots, x_m\} \text{ and}$$
$$w(r_i' - 1) > n_i, \quad i = 1, \dots, m-1.$$

$r_i' \in F$, and the bracket $(\dots)$ denotes the commutator (inductively defined). Then the pro-p group $G = F/\langle r_1, \dots, r_{m-1} \rangle$ has cohomological dimension 2.

If we choose $r_i' = 1$ for all i, then $w(r_i' - 1) = w(0) = \min \emptyset = \infty > n_i$, and hence for all primes p the pro-p group $G = F/\langle r_1, \dots, r_{m-1} \rangle$ has cohomological dimension 2 [20].

DEFINITION 5.8. *Let $F = \langle x_1, \dots, x_m \rangle$ be the free group on m generators. A group $G = F/\langle r_1, \dots r_{m-1} \rangle$ will be called of Koch type, if*

$$r_i = (x_m x_i a_{3i} \dots a_{n_i i}) \text{ with } a_{3i}, \dots a_{n_i i} \in \{x_1, \dots, x_{m-1}\} \text{ for } i = 1, \dots m-1.$$

Note that we consider discrete groups as well as pro-p groups in this context.
We will prove the following theorem:

THEOREM 5.9. *Discrete groups of Koch type are mild.*

Proof:
We have the following theorem, which is due to Anick [2]:

THEOREM 5.10. *Let $r = \{r_1, \dots, r_t\}$ be an arbitrary subset of the free group $F = \langle x_1, \dots, x_m \rangle$. Then r is mild if and only if the corresponding set $\rho = \{\rho_1, \dots, \rho_t\}$ of homogeneous elements in $U(gr(F))$ (see [1]) is strongly free.*

So to show that $\{r_1, \dots, r_{m-1}\} \in F$ is mild, it suffices to show that the corresponding set $\{\rho_1, \dots, \rho_{m-1}\} \in U(gr(F))$ is strongly free.
Recall the definition of a strongly free set in a connected graded $\mathbb{Z}$-algebra. For further details on this notion and on the definition of a strongly free set in a connected graded $\mathbb{Z}$-algebra refer to Anick [2].

DEFINITION 5.11. *A set $\{\rho_1, \dots, \rho_t\}$ of homogeneous elements in a connected graded $\mathbb{Z}$-algebra A is strongly free if and only if, for every prime p, its image in $A \otimes \mathbb{Z}/p\mathbb{Z}$ forms a strongly free set in $A \otimes \mathbb{Z}/p\mathbb{Z}$.*

Hence to show that $\{\rho_1, \ldots, \rho_{m-1}\}$ is strongly free in $U(gr(F))$, we have to show that the image of $\{\rho_1, \ldots, \rho_{m-1}\}$ in $U(gr(F)) \otimes \mathbb{Z}/p\mathbb{Z}$ is strongly free for all primes p. The image of $\{\rho_1, \ldots, \rho_{m-1}\}$ in $U(gr(F)) \otimes \mathbb{Z}/p\mathbb{Z}$ is $\{\rho_1 \otimes 1, \ldots, \rho_{m-1} \otimes 1\}$.
Now $gr(F)$ is the free Lie algebra on $\{x_1F_2, \ldots, x_mF_2\}$ over $\mathbb{Z}$ (see for instance Labute [23]). Hence the universal enveloping algebra $U(gr(F))$ is generated as a unital algebra by $\{x_1F_2, \ldots, x_mF_2\}$ and $U(gr(F)) \cong \mathbb{Z}\langle x_1F_2, \ldots, x_mF_2\rangle$ [7, Chapter II, 3.1].
As usual we write $\xi_i = x_iF_2$. Then $\{\xi_1, \ldots, \xi_m\}$ forms a basis of $U(gr(F))$. And hence $\{\xi_1 \otimes 1, \ldots, \xi_m \otimes 1\}$ is a basis of the free $\mathbb{Z}/p\mathbb{Z}$-algebra $U(gr(F)) \otimes \mathbb{Z}/p\mathbb{Z}$. (Note: $(\xi_i \otimes 1)(\xi_j \otimes 1) = \xi_i\xi_j \otimes 1$.)
Now Anick [1] has discovered a sufficient condition for a set of homogeneous elements in a free algebra to be strongly free. To state it we first need to define the term combinatorially free:

DEFINITION 5.12. *Let $\{\alpha_1, \ldots, \alpha_t\}$ be any set of nonzero monomials in the free associative algebra $\mathbb{Z}\langle\xi_1, \ldots, \xi_m\rangle$. Then $\{\alpha_1, \ldots, \alpha_t\}$ is combinatorially free if*

1. *no α_i is a submonomial of any α_j for $i \neq j$.*
2. *no α_i overlaps with any α_j.*

THEOREM 5.13 (Anick's Criterion)**.** *Let $\{\alpha_1, \ldots, \alpha_t\}$ be any set of nonzero homogeneous elements in the free associative algebra $\mathbb{Z}\langle\xi_1, \ldots, \xi_m\rangle$. Then $\{\alpha_1, \ldots, \alpha_t\}$ is strongly free in $\mathbb{Z}\langle\xi_1, \ldots, \xi_m\rangle$ if the leading terms of the α_i are combinatorially free.*

Now if the coefficients of the leading terms of $\rho_1, \ldots, \rho_{m-1}$ are ± 1, then the leading terms of the images of $\rho_1, \ldots, \rho_{m-1}$ in $U(gr(F)) \otimes \mathbb{Z}/p\mathbb{Z}$ coincide in the first component for all p. Moreover the leading terms of $\rho_1 \otimes 1, \ldots, \rho_{m-1} \otimes 1$ are combinatorially free if and only if so are the leading terms of $\rho_1, \ldots, \rho_{m-1}$.
Now if we take $r_i' = 1$ and $x_m \notin \{a_{3i}, \ldots, a_{n_i i}\}$ for all i, then $\rho_1, \ldots, \rho_{m-1}$ are strongly free:
For $r_i = (x_m x_i a_{3i} \ldots a_{n_i i}) \in F$ we obtain

$$\rho_i = r_i F_{n_i+1} \in gr(F) \subseteq U(gr(F)) \cong \mathbb{Z}\langle\xi_1, \ldots, \xi_m\rangle.$$

Consider the ordering $\xi_m > \xi_{m-1} > \ldots > \xi_1$.
Then $\rho_i = r_iF_{n_i+1} = (x_m x_i a_{3i} \ldots a_{n_i i})F_{n_i+1} = [x_mF_2, x_iF_2, \ldots, a_{n_i i}F_2] = [\xi_m, \xi_i, \ldots, \xi_{n_i i}] = \xi_m\xi_i \ldots \xi_{n_i i} +$ lower terms.
Note: $(\ ,\ )$ denotes the commutator in F and $[\ ,\ ]$ denotes the Lie bracket in $U(gr(F))$.
Hence the discrete group $G = F/\langle r_1, \ldots, r_{m-1}\rangle$ is mild, as initially claimed. □

COROLLARY 5.14. *Groups of Koch type have enough nilpotent torsion free quotients.*

This follows immediately from Theorem 5.4 on page 36.

Chapter 6

Cohomological Completeness

We will come to the notion of cohomological completeness now. It is one of the properties a group must have to belong to the class $\mathcal{F}$. In the first section we will introduce the notion of cohomological dimension and give some basic properties. For further details consider Wilson [42], Rotman [38] or Weibel [41]. In the second section we discuss the advantage of considering groups of low cohomological dimension. This is not a major restriction, since we are interested in torsion free groups anyway. We will give a useful criterion for a group of finite cohomological dimension to be cohomologically complete. Applying this criterion in the next section we prove that primitive link groups are cohomologically complete after having discussed the claim of Hillman et al that this holds for all link groups. In the last section we discuss what one can say about mild groups with respect to cohomological completeness.

6.1 Cohomological Dimension

DEFINITION 6.1. *A group G has cohomological dimension $\leq r$, if for all G-modules M we have $H^n(G, M) = 0$ for $n > r$. We write $cd(G) \leq r$. We say G has cohomological dimension r, if $cd(G) \leq r$ but not $cd(G) \leq r - 1$. We then write $cd(G) = r$.*

This is equivalent to saying: there exists a projective resolution $P \xrightarrow{\epsilon} \mathbb{Z}$ of $\mathbb{Z}G$-modules such that $P_i = 0$ for $i > r$. It is obvious from the definition that $\mathrm{cd}(G) \in \mathbb{N} \cup \infty$.
Some basic properties of cohomological dimension are the following:

LEMMA 6.2. *Let H be a subgroup of a group G. Then $cd(H) \leq cd(G)$*

LEMMA 6.3. *If $cd(G) < \infty$, then G is torsion free.*

We will define the notion of a p-torsion module to be able to define cohomological-p-dimension:

DEFINITION 6.4. *A G-module M is said to be a p-torsion module, if every element m is of order p^{n_m} for some $n_m \in \mathbb{N}$.*

DEFINITION 6.5. *For a group G the cohomological p-dimension $cd_p(G)$ is defined to be:*

$$\sup\{n \mid \exists \textit{ a discrete p-torsion } G-\textit{module } A \textit{ with } H^n(G, M) \neq 0\}$$

The following holds (see for instance Wilson [42]):

LEMMA 6.6. *Let G be a group. Then $cd_p(G) < n$ if and only if $H^n(G, M) = 0$ for all simple G-modules M satisfying $pM = 0$.*

For a pro-p-group we even obtain:

LEMMA 6.7. *Let G be a pro-p-group. Then we have $cd_p(G) < n$ if and only if $H^n(G, \mathbb{Z}/p\mathbb{Z}) = 0$.*

Using the definition of cohomological p-dimension we can restate the definition of cohomological dimension in the following way:

$$\mathrm{cd}(G) = \sup\{\mathrm{cd}_p(G) \mid p \text{ prime}\}$$

6.2 Groups with Low Cohomological Dimension

Labute [22] gives the following example. He proves that cohomological completeness of a one-relator group G holds if and only if $cd(\hat{G}^p) \leq 2$. He deduces the equivalence by using results of Lyndon [32] on one-relator groups.
Initially we thought we could extend the result to groups G with $cd(G) = 2$. However this is not possible as a counterexample of Baumslag, Dyer and Heller [5] shows:
They give an example of a group that has cohomological dimension 2 whereas the cohomology groups of its pro-p-completions are all trivial and so G is not cohomologically complete.
It is not clear, whether the following holds: If G is a discrete group with $cd_p(G) = 2$, then G is cohomologically complete, if and only if $cd(\hat{G}^p) = 2$ for all p.
Obviously one direction holds: if G is cohomologically complete, then $\hat{G}^p$ has cohomological dimension 2 for all p. Moreover the canonical homomorphism:

$$H^n(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) \to H^n(G, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism for $n = 0, 1$, and is injective for $n = 2$ for all primes p (refer to Linnell-Schick [28] and Serre [40]).
Hence to show that an arbitrary group G is cohomologically complete, it suffices to show that the above homomorphism is surjective for $n = 2$ and bijective for $n > 2$. So it seems to be easier, to work with groups of low cohomological dimension. In particular, if G has cohomological dimension 2, one only has to check whether the homomorphism is surjective for $n = 2$. This follows by the following lemma for $n = 2$.

LEMMA 6.8. *Let G be a discrete group. If $cd(G) = k$ and the homomorphism*

$$H^n(\hat{G}^p, M) \to H^n(G, M)$$

is surjective for $n \leq k$ for all finite discrete $\hat{G}^p$-modules M, then G is cohomologically complete.

Proof: Suppose $cd(G) = k$ and $H^n(\hat{G}^p, M) \to H^n(G, M)$ is surjective for $n \leq k$ for all finite discrete $\hat{G}^p$-modules M. Then it is surjective for all n, since $H^n(G, M) = 0$ for $n > k$. By a remark of Serre [40, Ex. 1, p.15], the homomorphism is hence bijective for all n, and so G is cohomologically complete. □
Note that in particular we do not need to verify the cohomological dimension of $\hat{G}^p$.

6.3 Are Link Groups Cohomologically Complete?

Recently Hillman, Matei and Morishita have claimed that link groups are cohomologically complete [17, Theorem 1.2.1]. However unfortunately their proof seems to have a gap.
They aim to show by induction on the length of the finite p-primary $\hat{G}^p$-module M that, if H is a finite subgroup of G of p-power index, then there is a smaller subgroup G_1 of finite p-power index, such that the restriction map

$$H^2(H, M) \to H^2(G_1, M)$$

is trivial. Applying the equivalence of Serre [40, Ex. 1, p.15] they then deduce the cohomological completeness of G.
However in the start of induction they denote the kernel of an epimorphism from G_1 to $\mathbb{Z}/p\mathbb{Z}$ with K, where they construct the epimorphism such that K has p-power index in

G_1. Using the theory of spectral sequences, in particular the Lyndon/Hochschild-Serre spectral sequence, they end up with the claim that the homomorphism

$$H^2(G_1, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$$

is the zero map.
The following counter example shows that this does not always hold.

6.3.1 Counter Example

Consider the link $L \bigcup_d S^1$ that is the disjoint union of another arbitrary link L and a separate copy of S^1. Then, applying the Seifert-Van Kampen Theorem, its link group is the free product of the link groups of its components $G = G_L \star \mathbb{Z}$.
Then we obtain the epimorphism $G = G_L \star \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$,
where the whole of G_L is mapped to 0 and $1_{\mathbb{Z}} \mapsto 1_{\mathbb{Z}/p\mathbb{Z}}$.
(Note that G is a subgroup of finite p-power index of itself.)
Denote the kernel of this epimorphism by K_1. Then $K_1 > K$ where $K = \star_{\mathbb{Z}} G_L$.
I. e. K is the kernel of the obvious epimorphism $G_L \star \mathbb{Z} \to \mathbb{Z}$.
Then

$$G_L \hookrightarrow K_1$$

$$\text{and hence we obtain} \quad G_L \hookrightarrow K \hookrightarrow K_1 \hookrightarrow G \to G_L$$

where the composition is the identity of G_L.
Hence by functoriality the composition of the induced maps in cohomology is the identity as well:

$$H^2(G_L) \to H^2(G) \to H^2(K_1) \to H^2(K) \to H^2(G_L)$$

I. e. the composition is in particular surjective and so if $H^2(G_L) \neq \{e\}$ none of the maps inbetween can be the zero map. We have $\mathrm{cd}(G_L) = 2$ and so $H^2(G_L) \neq \{e\}$.
However this means that in particular $H^2(G) \to H^2(K_1)$, which is the restriction map, (since it is induced by the injection $K_1 \hookrightarrow G$,) is not trivial.

[To be more precise, we know the following: Since the above composition of maps in cohomology is the identity, the first map must be injective. However since cohomology is additive $H^2(G) = H^2(G_L) \oplus \underbrace{H^2(\mathbb{Z})}_{=\{e\}} = H^2(G_L)$, the map $H^2(G) \to H^2(K_1)$, which comes down to $H^2(G_L) \to H^2(K_1)$, is injective. I. e. we know that the restriction map $H^2(G) \to H^2(K_1)$ is not trivial and moreover is injective.]

6.3.2 A Closer Look

We will have a closer look on the proof of Hillman, Matei and Morishita.
They consider the Lyndon/Hochschild-Serre spectral sequence for H as an extension of $\mathbb{Z}/p\mathbb{Z}$ by K with $\mathbb{E}_2$-term

$$E_2^{s,t} = H^s(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})).$$

They claim that since $\mathbb{Z}/p\mathbb{Z}$ has cohomological period 2, they obtain isomorphisms

$$\gamma_2^{s,t} : E_2^{s,t} \to E_2^{s+2,t} \text{ for all } s, t \geq 0.$$

Indeed, $\mathbb{Z}/p\mathbb{Z}$ has cohomological period 2, and we have

$$\begin{aligned} H^0(\mathbb{Z}/p\mathbb{Z}, M) &= M^{\mathbb{Z}/p\mathbb{Z}} \\ H^n(\mathbb{Z}/p\mathbb{Z}, M) &= H^{n+2}(\mathbb{Z}/p\mathbb{Z}, M) \quad \text{for } n \text{ even, } n > 0 \\ H^n(\mathbb{Z}/p\mathbb{Z}, M) &= H^{n+2}(\mathbb{Z}/p\mathbb{Z}, M) \quad \text{for } n \text{ odd} \end{aligned}$$

or more precisely for $G = \mathbb{Z}/p\mathbb{Z}$:

$$\begin{aligned} H^0(G, M) &= M^G \\ H^n(G, M) &= M^G/NM \quad \text{for } n \text{ even }, \ n \geq 2 \end{aligned}$$

with $N = 1 + g + g^2 + \ldots + g^{p-1}$ where g is a generator of $\mathbb{Z}/p\mathbb{Z}$
(e.g. Evens [13, p.6], or Weibel [41, p.168]).
In our case this comes down to:

$$\begin{aligned} H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) &= (H^t(K, \mathbb{Z}/p\mathbb{Z}))^{\mathbb{Z}/p\mathbb{Z}} \\ H^n(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) &= (H^t(K, \mathbb{Z}/p\mathbb{Z}))^{\mathbb{Z}/p\mathbb{Z}}/NH^t(K, \mathbb{Z}/p\mathbb{Z}) \quad \text{for } n \text{ even }, n \geq 2 \end{aligned}$$

Thus we obtain isomorphisms

$$H^s(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) \to H^{s+2}(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) \quad \text{for } s \geq 1, \ t \geq 0.$$

what comes down to isomorphisms

$$E_2^{s,t} \cong E_2^{s+2,t} \quad \text{for } s \geq 1, \ t \geq 0$$

However in general $H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) \neq H^2(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z}))$ and hence in contrast to the claim of Hillman et al in general $E_2^{0,t} \ncong E_2^{2,t}$.

In the sequel of their proof Hillman et al use exactly the isomorphism $E_2^{0,2} \to E_2^{2,2}$ to deduce that $E_\infty^{0,2} = 0$, which implies that the edge homomorphism from $H^2(H, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$, that factors through $E_\infty^{0,2}$, must be zero. Finally they deduce that hence the restriction map $\mathrm{res}_K^H : H^2(H, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$, which induces the edge homomorphism, must be zero as well.

For further details on spectral sequences and the notions used here refer to McCleary [33], Ribes-Zalesskii [36] or Weibel [41].

However in general there is no isomorphism

$$H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) \to H^2(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z}))$$

and hence one cannot deduce that the restriction map $\mathrm{res}_K^H : H^2(H, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$ is zero.

So in general the proof seems to have a gap, however maybe there are particular cases for which there is indeed an isomorphism $H^0(\mathbb{Z}/p\mathbb{Z}, H^2(K, \mathbb{Z}/p\mathbb{Z})) \cong H^2(\mathbb{Z}/p\mathbb{Z}, H^2(K, \mathbb{Z}/p\mathbb{Z}))$

Considering the above, we see that this is the case if and only if $NH^2(K, \mathbb{Z}/p\mathbb{Z}) = 0$.

Now

$$\begin{aligned} NH^2(K, \mathbb{Z}/p\mathbb{Z}) &= \{(1 + g + g^2 + \ldots + g^{p-1})\bar{\phi} \mid \bar{\phi} = \phi \mathrm{im} d_2' \in \mathrm{ker} d_3' / \mathrm{im} d_2' = H^2(K, \mathbb{Z}/p\mathbb{Z})\} \\ &= \{\bar{\phi} + g\bar{\phi} + g^2\bar{\phi} + \ldots + g^{p-1}\bar{\phi} \mid \bar{\phi} \text{ as above}\} \end{aligned}$$

[Note that we consider $N \xrightarrow{\epsilon} \mathbb{Z}$ to be a projective $\mathbb{Z}K$-module resolution of $\mathbb{Z}$ here with module homomorphisms $d_i : N_i \to N_{i-1}$.]

So we would like to investigate the module map

$$\mathbb{Z}/p\mathbb{Z} \times H^2(K, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$$

more in detail. It does not seem to be possible to find any general pattern of this map, and it seems unlikely that it is the trivial module map for a huge class of groups. What we can say is the following:

If K is a central subgroup of H, then $H/K = \mathbb{Z}/p\mathbb{Z}$ acts trivially on $H^2(K, \mathbb{Z}/p\mathbb{Z})$ [41, Example 6.8.4].

So let us assume K is a central subgroup of H. Then $(H^2(K, \mathbb{Z}/p\mathbb{Z}))^{\mathbb{Z}/p\mathbb{Z}} = H^2(K, \mathbb{Z}/p\mathbb{Z})$.

Moreover

$$\begin{aligned} NH^2(K, \mathbb{Z}/p\mathbb{Z}) &= \{\bar{\phi} + g\bar{\phi} + g^2\bar{\phi} + \ldots + g^{p-1}\bar{\phi} \mid \bar{\phi} \text{ as above}\} \\ &= \{\underbrace{\bar{\phi} + \bar{\phi} + \ldots + \bar{\phi}}_{p \text{ summands}} \mid \bar{\phi} \text{ as above}\} \\ &= 0 \end{aligned}$$

Note that $\phi \in \ker d_3' \subseteq \mathrm{Hom}_{\mathbb{Z}K}(N_2, \mathbb{Z}/p\mathbb{Z})$, so $\phi : N_2 \to \mathbb{Z}/p\mathbb{Z}$, thus adding ϕ p times gives the zero map.
Hence if K is a central subgroup of H

$$H^0(\mathbb{Z}/p\mathbb{Z}, H^2(K, \mathbb{Z}/p\mathbb{Z})) = H^2(\mathbb{Z}/p\mathbb{Z}, H^2(K, \mathbb{Z}/p\mathbb{Z}))$$

and so the gap in the proof could be fixed if additionally we require K to be a central subgroup of H.
In the counterexample we constructed, K is a free product and so in particular is not abelian and so cannot be contained in the centre of H.
This additional requirement is quite strong, since it says in particular that K must be abelian, and moreover the centre $Z(H) > K$. Since K has index p in H, the centre of H must be of index $\leq p$. On the other hand the centre of a group has never prime index, so $Z(H) = H$ and H is abelian. So the induction hypothesis must be reformulated: Suppose G has an abelian subgroup H of finite p-power index, then there is a subgroup $K \leq H$ of finite p-power index, such that $\mathrm{res}^H_K = 0$.
At the end of the proof, the subgroup H, which will be needed, is the group G itself. So this means in particular, that the proof is fixed only for the case G is abelian. For a link group it seems very unlikely to be abelian. (Of course we know $\mathbb{Z}$ is an abelian link group.)
In the Appendix we demonstrate an extension of the proof to abelian groups G with $cd(G) = 2$ that either admit a surjection to $\mathbb{Z}$ or imbed injectively in their pro-p completions. It is however not clear, whether the strong requirements apply to a large class of groups.

6.4 Primitive Link Groups

So unfortunately there seems to be a gap in the proof of Hillman et al and it is still an open question whether link groups are cohomologically complete. Using a different approach, we can show that primitive link groups are cohomologically complete. Recall from Section 3.2 that a link group is primitive, if the linking diagram of its associated link is connected mod p for every prime p.

THEOREM 6.9. *Primitive link groups are cohomologically complete.*

Proof:
Suppose G is a primitive link group with associated link L. Then we want to show that

$$H^n(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) \to H^n(G, \mathbb{Z}/p\mathbb{Z}) \tag{6.1}$$

is an isomorphism for every $n \in \mathbb{Z}$.
We know that the map in (6.1) is an isomorphism for $n = 0, 1$ and is injective for $n = 2$ [28, Lemma 4.5]. Moreover we know that the classifying space of G is the 3-dimensional manifold $S^3 \setminus U(L)$ (which is homotopy-equivalent to a 2-dimensional CW-complex), where $U(L)$ is an open tubular neighbourhood of L [28, Proposition 5.34].
Now Linnell and Schick could show the following [29, Proposition 2.4]

PROPOSITION 6.10. *Let G be a discrete group with classifying space of dimension 2. Then G is cohomologically complete, if and only if*

1. $dim_{\mathbb{Z}/p\mathbb{Z}} H^2(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) = dim_{\mathbb{Z}/p\mathbb{Z}} H^2(G, \mathbb{Z}/p\mathbb{Z})$ *and*
2. $R/\overline{[R, R]}$ *is a free* $\mathbb{Z}_p[[\hat{G}^p]]$*-module*

where $\hat{G}^p = F/R$, with F a finitely generated free pro-p group and the module map is induced by conjugation (it is described in detail in Section 3.1.1 on page 22).

In the sequel of the same paper they prove, using a result of Milnor [25] on presentations of the descending central series quotients of link groups, that if G is a link group with d components, then $\hat{G}^p$ has a presentation with d generators and $d - 1$ relations. Using this and some cohomology theory they deduce that for G a primitive link group

$$\dim_{\mathbb{Z}/p\mathbb{Z}} H^2(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) = d - 1 = \dim_{\mathbb{Z}/p\mathbb{Z}} H^2(G, \mathbb{Z}/p\mathbb{Z}) \tag{6.2}$$

So in order to prove that primitive link groups are cohomologically complete, it remains to establish that

$$R/\overline{[R, R]} \text{ is a free } \mathbb{Z}_p[[\hat{G}^p]]\text{-module.} \tag{6.3}$$

Now Labute shows that this holds for mild pro-p groups [26, Theorem 1.2]. So to show that G is cohomologically complete, it suffices to show that $\hat{G}^p$ is mild.

LEMMA 6.11. *Let G be a primitive link group. Then $\hat{G}^p$ is mild for all primes p.*

Proof: Let L be a free $\mathbb{Z}$-Lie algebra on d generators, then there is a canonical map $L \to gr(F)$. Note that $L = gr(\Gamma_d)$, where Γ_d is the discrete free group on d generators ([23]), and hence L imbeds naturally in $gr(F)$, where $gr(F)$ coincides with the free $\mathbb{Z}_p$-module on d generators. Now let $r_1, \dots, r_s \in F$ generate a closed normal subgroup R and let $\rho_1, \dots, \rho_s$ be their initial forms (see Section 3.1 on page 21). Suppose $\{\rho_1, \dots, \rho_s\} \subset L \subset gr(F)$. Then let $\mathfrak{r}$ be the ideal of L, that is generated by $\{\rho_1, \dots, \rho_s\}$ and let $\mathfrak{R}$ be the (closed) ideal of $gr(F)$ that is generated by $\{\rho_1, \dots, \rho_s\}$.
Linnell and Schick have shown the following [29, Lemma 2.14]:

LEMMA 6.12. *If $L/\mathfrak{r}$ is a free $\mathbb{Z}$-module, then $gr(F)/\mathfrak{R}$ is a free $\mathbb{Z}_p$-module.*

Now $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a $U(L/\mathfrak{r})$-module, where $U(L/\mathfrak{r})$ is the universal enveloping algebra of $L/\mathfrak{r}$. For more details on the module map refer to Section 3.1.1 on page 22. Similarly $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$ is a $U(gr(F)/\mathfrak{R})$-module. If $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free $U(L/\mathfrak{r})$-module with a finite basis, then $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$ is a free $U(gr(F)/\mathfrak{R})$-module with the same basis (only considered mod $\overline{[\mathfrak{R},\mathfrak{R}]}$ of course), since $U(gr(F)/\mathfrak{R})$ is degree-wise the pro-p completion of $U(L/\mathfrak{r})$ and $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is degree-wise the pro-p completion of $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$. By [29, Lemma 2.15] the $U(L/\mathfrak{r})$-module multiplication:

$$U(L/\mathfrak{r}) \times \mathfrak{r}/[\mathfrak{r},\mathfrak{r}] \to \mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$$

extends by continuity to the $U(gr(F)/\mathfrak{R})$-module multiplication:

$$U(gr(F)/\mathfrak{R}) \times \mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]} \to \mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$$

Suppose finally that $\hat{G}^p = F/R$ is the pro-p completion of a primitive link group G of a d-component link. Then, as we have mentioned, there exists a specific presentation of $\hat{G}^p$ with $d-1$ relations $r_1, \ldots, r_{d-1}$. Moreover this presentation is given such that the initial forms $\rho_1, \ldots, \rho_{d-1} \subset L \subset gr(F)$. Since the associated link is primitive, i. e. its linking diagram is connected mod p for every prime p, according to a result of Labute [25, Theorem 2], $L/\mathfrak{r}$ is a free $\mathbb{Z}$-module and $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free $U(L/\mathfrak{r})$-module, generated by the images of $\rho_1, \ldots, \rho_{d-1}$ in $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$. Hence we obtain the following lemma:

LEMMA 6.13. *$gr(F)/\mathfrak{R}$ is a free $\mathbb{Z}_p$-module, generated by the images of $\rho_1, \ldots, \rho_{d-1}$ in $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$, (see Lemma 6.12), and $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$ is a free $U(gr(F)/\mathfrak{R})$-module generated by the images of $\rho_1, \ldots, \rho_{d-1}$ in $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$.*

By the very definition of strong freeness, the conditions $gr(F)/\mathfrak{R}$ is a free $\mathbb{Z}_p$-module and $\mathfrak{R}/\overline{[\mathfrak{R},\mathfrak{R}]}$ is a free $\mathbb{Z}_p[[\hat{G}^p]]$-module on the images of $\rho_1, \ldots, \rho_d$ say that the sequence $\rho_1, \ldots, \rho_d$ is strongly free [refer to Section 3.3].
However this in turn implies that $\hat{G}^p = F/R$ is mild (with respect to the lower central series). This finishes the proof of Lemma 6.11. □
Applying Theorem 2.1 of [26] it follows immediately that $R/\overline{[R,R]}$ is a free $\mathbb{Z}_p[[\hat{G}^p]]$-module on the images of the r_i. Labute has used his proof of Theorem 2.1 in different settings [24, Theorem 1 and 3], [23, Section 3] and [22, Section 3] and after having studied them thoroughly, it is obvious that the results of Theorem 2.1 [26] hold as well for pro-p groups that are mild with respect to the lower central series. This completes the proof of Theorem 6.9 and shows that G is cohomologically complete. □

For the record an alternative version of the proof that $R/\overline{[R,R]}$ is a free $\mathbb{Z}_p[[\hat{G}^p]]$-module (6.3) is given, without using the notion of a mild group. Hence the conditions of the pro-p version of the following theorem [24, Theorem 1] hold.

THEOREM 6.14. *If $L/\mathfrak{r}$ is a free $\mathbb{Z}$-module and $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free $U(L/\mathfrak{r})$-module on the images of the ρ_i, then $gr(G) = L/\mathfrak{r}$, where we have a discrete setting here.*

It remains to verify that these conditions, in the pro-p world as well, imply $\mathfrak{R} = gr(R)$. On the bottom of the page, Labute remarked that it is possible to transfer the theorem to the pro-p world.
The proof is exactly the same as the proof of Theorem 1 [24], where subgroups are replaced by closed subgroups, $\mathbb{Z}$ is replaced by $\mathbb{Z}_p$ and the group ring $\mathbb{Z}[G]$ is replaced by the completed group algebra $\mathbb{Z}_p[[\hat{G}^p]]$. Note that virtually the same proof can also be found in [26, Proof of Theorem 1.2][23, Section 3] and [22, Section 3].
We will not copy the proof, but explain the idea of the proof and point out where to pay particular attention to in our case.
Let $R_n = R \cap F_n$. Then this gives a filtration on R, and we obtain a graded $\mathbb{Z}_p$-Lie algebra $gr(R)$. Now to be more precise, we have

$$gr(R) = \bigoplus_{n\in\mathbb{N}} gr_n(R) = \bigoplus_{n\in\mathbb{N}} R_n/R_{n+1} = \bigoplus_{n\in\mathbb{N}} (R \cap F_n)/(R \cap F_{n+1})$$

Thus we obtain the following embedding:

$$gr(R) \to gr(F)$$

given by $\sum_{n\in\mathbb{N}} s_n(R \cap F_{n+1}) \mapsto \sum_{n\in\mathbb{N}} s_n F_{n+1}$. If we identify $gr(R)$ with its image in $gr(F)$, then $\mathfrak{R}$ embeds naturally in $gr(R)$: $\mathfrak{R}$ is generated as a closed ideal by $r_1F_{k_1}, \dots, r_dF_{k_d}$, with $r_1, \dots, r_d$ generating R. So $r_i \in F_{k_i}$ implies $r_i \in R \cap F_{k_i}$ and so $r_iF_{k_i} \in gr(R)$. Hence $\mathfrak{R} \subseteq gr(R)$. Note that we assume all subgroups to be closed.
We therefore obtain a homomorphism

$$\theta : \mathfrak{R}/[\mathfrak{R},\mathfrak{R}] \to gr(R)/[gr(R), gr(R)].$$

Obviously θ is bijective if and only if $\mathfrak{R} = gr(R)$. [This follows by using induction on the degrees.]
Hence in order to show $\mathfrak{R} = gr(R)$ it suffices to show that θ is bijective.
Now let $M = R/[R,R]$. Then we obtain a filtration (M_n) of M, where M_n is the image of R_n in M under the canonial projection. Then $gr(M) = gr(R)/gr([R,R])$, where $[R,R]_n$ is obtained analogously to R_n.

In the sequel the auxiliary homomorphism

$$\theta' : gr(R)/[gr(R), gr(R)] \to gr(M)$$

is constructed. It is defined by

$$\sum_{n\in\mathbb{N}} s_n(R \cap F_{n+1})[gr(R), gr(R)] \mapsto \sum_{n\in\mathbb{N}} s_n(R \cap F_{n+1})gr([R, R]).$$

Next we want to show inductively on the degrees that θ and θ' are bijective. We assume the bijectivity of degree $< k$ holds. We hence aim to establish the bijectivity of degree k. The bijectivity of θ' is obtained exactly as in [23, p. 20] and [26, p. 19]. Moreover bijectivity of θ in degree $< k$ implies injectivity in degree k. It remains to establish the surjectivity of θ. To prove this, it suffices to show that $\theta'' = \theta' \circ \theta$ is surjective in degree k. As a result in the proof we obtain $gr(M)$ is a free $gr(\mathbb{Z}_p[[\hat{G}^p]])$-module on the images of $r_i, \ldots, r_d$. However this implies that $M = R/[R, R]$ is a free $\mathbb{Z}_p[[\hat{G}^p]]$-module on the images of $r_1, \ldots, r_d$.
Finally this asserts that G is cohomologically complete. □

6.5 Mild Groups

Anick initially hoped that he could show that all discrete mild groups have cohomological dimension 2. He did not succeed, however found two counterexamples [2]. So we know that not all discrete mild groups have cohomological dimension 2, but there is good hope that there are a lot of mild groups with cohomological dimension 2. Note that all mild pro-p groups have indeed cohomological dimension 2 [26, Theorem 1.2]. We can certainly say:

THEOREM 6.15. *Let G be a discrete group. If G is mild and residually a finite p-group, then $cd(G) = 2$.*

Proof:
Suppose G is mild and residually a finite p-group for some prime p. To show that $cd(G) = 2$ it suffices to verify that

$$0 \to R/[R, R] \to \mathbb{Z}G^m \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}G$-modules.
Here ε is the augmentation map, $\mathbb{Z}G^m$ is a free $\mathbb{Z}G$-module and the sequence is constructed to be exact. So it remains to show that $R/[R, R]$ is a free $\mathbb{Z}G$-module.

We know that $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free $U(gr(F)/\mathfrak{r})$-module on the images of the ρ_i, and $U(gr(F)/\mathfrak{r}) \cong gr(\mathbb{Z}G)$ if G is mild [24]. The grading of $gr(\mathbb{Z}G)$ is induced by the powers of the augmentation ideal $I(G)$ of $\mathbb{Z}G$. Moreover $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}] \cong gr(M)$ with $M = R/[R,R]$.
[Note that

$$gr(M) = \bigoplus_{n\geq 0} M_n/M_{n+1} = \bigoplus_{n\geq 0}(R \cap F_n)[R,R]/(R \cap F_{n+1})[R,R]$$

and the isomorphism is given by $\rho_i[\mathfrak{r},\mathfrak{r}] = r_i F_{n_i+1}[\mathfrak{r},\mathfrak{r}] \mapsto r_i[R,R]M_{n_i+1}$.]
So $gr(M)$ is a free $gr(\mathbb{Z}G)$-module on the images of the $\rho_i[\mathfrak{r},\mathfrak{r}]$, i.e. on

$$r_1[R,R]M_{n_1+1}, \ldots, r_{m-1}[R,R]M_{n_{m-1}+1}.$$

To show that M is a free $\mathbb{Z}G$-module on $r_1[R,R]\ldots, r_{m-1}[R,R]$, it suffices to show that

$$\gamma_1 r_1[R,R] + \ldots + \gamma_{m-1} r_{m-1}[R,R] = 0, \quad \gamma_i \in \mathbb{Z}G$$

implies $\gamma_i = 0$ for all i.
Now "move" the relation $\gamma_1 r_1[R,R] + \ldots + \gamma_{m-1} r_{m-1}[R,R] = 0$ to $gr(M)$ and $gr(\mathbb{Z}G)$:
We obtain

$$\bar{\gamma}_1 \bar{r}_1 + \ldots + \bar{\gamma}_{m-1} \bar{r}_{m-1} = 0$$

with $\bar{\gamma}_i \in gr_0(\mathbb{Z}G) = \mathbb{Z}G/I(G)$ and $\bar{r}_i \in gr(M)$, such that the $\bar{r}_i$ are the basis elements of the free module $gr(M)$ over $gr(\mathbb{Z}G)$, i.e. $\bar{r}_i = r_i[R,R]M_{n_i+1}$.
Since $gr(M)$ is a free $gr(\mathbb{Z}G)$-module on $\bar{r}_1, \ldots, \bar{r}_{m-1}$, we have $\bar{\gamma}_i = 0$ for all i. However $\bar{\gamma}_i = \gamma_i I(G)$, i.e. $\bar{\gamma}_i = 0 \iff \gamma_i \in I(G)$.
Knowing this, we can "move" the γ_i to the next level in $gr(\mathbb{Z}G)$, i.e. to $gr_1(\mathbb{Z}G) = I(G)/I(G)^2$.
We obtain $\bar{\gamma}_1 \bar{r}_1 + \ldots + \bar{\gamma}_{m-1} \bar{r}_{m-1} = 0$ with $\bar{r}_i$ as above and $\bar{\gamma}_i = \gamma_i I(G)^2 \in gr_1(\mathbb{Z}G)$.
By the same argument as above, $\bar{\gamma}_i = 0$ which means $\gamma_i \in I(G)^2$.
Continuing like this, we obtain

$$\gamma_i \in \bigcap_{n\geq 0} I(G)^n$$

So if $\bigcap_{n\geq 0} I(G)^n = 0$, all $\gamma_i = 0$ and we are done, i.e. M is a free $\mathbb{Z}G$-module.
Now by assumption G is residually a finite p-group, i.e. for every $g \in G \setminus \{e\}$ there is a normal subgroup H_g of finite p-power index such that $gH_g \neq H_g$. Denote the set of normal subgroups of G of finite p-power index by I.
Thus if we consider the canonical homomorphism:

$$j : G \to \hat{G}^p$$

we know $\ker(j) = \bigcap_{H \in I} H$ (see Wilson [42, Proposition 1.4.4]). Now for every $g \in G \backslash \{e\}$, there is a normal subgroup $H_g \lhd G$ of finite p-power index, i.e. $H_g \in I$ with $g \notin H_g$. Hence

$$\bigcap_{H \in I} H = \{e\} \text{ and so}$$

$$\ker(j) = \{e\}$$

i.e. j is injective and so G imbeds injectively in $\hat{G}^p$.
Now consider the augmentation ideal $I(\hat{G}^p) \subseteq \mathbb{Z}_p[[\hat{G}^p]]$ of the completed group algebra.

$$\bigcap_{n \geq 0} I(\hat{G}^p)^n = 0 \quad \text{(this equality always holds for pro-}p\text{-groups.)}$$

Now $I(G) \hookrightarrow I(\hat{G}^p)$, since $G \hookrightarrow \hat{G}^p$, and hence

$$\bigcap_{n \geq 0} I(G)^n \hookrightarrow \bigcap_{n \geq 0} I(\hat{G}^p)^n \text{ i.e. } \bigcap_{n \geq 0} I(G)^n = 0$$

Hence $\gamma_i = 0$ for $i = 1, \ldots, m-1$
$\Rightarrow M$ is a free $\mathbb{Z}G$ -module
$\Rightarrow \mathrm{cd}(G) = 2$. □

The result can be applied for instance to groups of Koch type (refer to Example 5.8 on page 39): pro-p groups of Koch type $G = F/\langle r_1, \ldots, r_{m-1} \rangle$ have cohomological dimension 2 (refer to Example 5.7 on page 39). However these groups are exactly the pro-p completions of the discrete group $G = F/\langle r_1, \ldots, r_{m-1} \rangle$ [42]. [If it is clear from the context we sometimes use the same notation for the discrete group as for the pro-p group, although these objects are in general not the same.]
For our purposes it makes sense to study groups of Koch type, since we know already that (discrete) groups of Koch type are mild and hence have enough nilpotent torsion free quotients (Corollary 5.14 on page 41). So we naturally aim to find examples that are cohomologically complete.
A necessary condition for cohomological completeness of the discrete group G of Koch type is that G has $cd(G) = 2$. The theorem gives a sufficient condition for $cd(G) = 2$.

Thus in order to show that these groups are cohomologically complete, it remains to show that the map $H^2(\hat{G}^p, \mathbb{Z}/p\mathbb{Z}) \to H^2(G, \mathbb{Z}/p\mathbb{Z})$ is surjective. In the above proof we showed that G imbeds injectively in its pro-p completions. So applying Theorem 8.4

we obtain that a discrete group G that is abelian, mild and residually a finite p-group, is cohomologically complete.

However as initially explained, there is a good chance, that there are more discrete mild (residually finite p-groups) that are cohomologically complete.

Chapter 7

The Class $\mathcal{F}$ of Groups

In the first section we will shortly state precisely what it means for a group to lie in $\mathcal{F}$. In the second section we will sum up our results on cohomological completeness and enough nilpotent torsion free quotients from the previous chapters to give new examples of groups that lie in $\mathcal{F}$. It is deduced that primitive link groups lie in $\mathcal{F}$. Finally in the third section we will show that for the Atiyah Conjecture to be true for all extensions of finite and elementary amenable quotient of a group in $\mathcal{F}$, it is sufficient to show that it holds for specific subgroups.

7.1 General Remarks

To be more precise than in the Introduction, a group G lies in the class $\mathcal{F}$ of groups if the following holds:

1. G has got a finite classifying space
2. G ist cohomologically complete
3. G has enough nilpotent torsion free quotients

As mentioned in the Introduction, the important and characteristic property of the class $\mathcal{F}$ of groups is that the Atiyah Conjecture is inherited by extensions with finite or elementary amenable quotient. I. e. if the Atiyah Conjecture holds for a group G in $\mathcal{F}$, then it also holds for all extensions of G with finite or elementary amenable quotient.
For further details on cohomological completeness refer to Section 2.2.
In the context of the strong Atiyah Conjecture, several classes of groups, satisfying specific requirements have been defined [31]. Note that, if a torsion free group G,

belongs to Linnell's class C of groups, i. e. is contained in the smallest class of groups, which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients, then the strong Atiyah Conjecture is true for G [31, Theorem 10.19]. So there is a large number of torsion free groups, for which the Atiyah Conjecture holds.
Consider the following definition:

DEFINITION 7.1. *A group G is said to be residually torsion free nilpotent, if it has a normal series $(H_i)_{i\in\mathbb{N}}$ of subgroups such that*

1. $\bigcap_{i\in\mathbb{N}} H_i = \{e\}$
2. *G/H_i is torsion free nilpotent for all $i \in \mathbb{N}$.*

For the class $\mathcal{F}$ of groups it is useful that Linnell and Schick have shown the following [28, Corollary 3.3]:

COROLLARY 7.2. *The strong Atiyah Conjecture is true for $\bar{\mathbb{Q}}G$, if G is residually torsion free nilpotent or residually torsion free solvable.*

In the previous chapters we have given a large supply on new examples of groups that are cohomologically complete and have enough nilpotent torsion free quotients.
In particular in Theorem 6.15 on page 53 we have shown that mild groups, that are residually finite p-groups have cohomological dimension 2. Actually there is good hope that there are a lot more discrete mild groups that have cohomological dimension 2, as we have explained. So there is a good chance that there are mild groups that are cohomologically complete. In Chapter 5 we established that mild groups have enough nilpotent torsion free quotients (Corollary 5.4 on page 36). Hence mild groups are an interesting well worth subject to study in order to find more groups that lie in $\mathcal{F}$.

7.2 Primitive Link Groups

Putting together our results from the previous chapters on primitive link groups, we obtain the following result:

THEOREM 7.3. *Let G be a primitive link group. Then G lies in $\mathcal{F}$.*

Proof:
Primitive link groups have a classifying space of dimension 2 [28].

As we have shown, primitive link groups have enough nilpotent torsion free quotients (Theorem 5.6 on page 38) and are cohomologically complete (Theorem 6.9 on page 49). This completes the proof. □

Using Corollary 7.2 we can deduce the following:

COROLLARY 7.4. *For all primitive link groups that are residually torsion free nilpotent or residually torsion free solvable, the Atiyah Conjecture and the Zero Divisor Conjecture also hold for group extensions with finite or elementary amenable quotient.*

7.3 Atiyah Conjecture for Subgroups in $\mathcal{F}$

As explained in Chapter 2 the characteristic property of the class $\mathcal{F}$ of groups is that the Atiyah Conjecture is inherited by extensions with finite or elementary amenable quotient. We will extend the result. If a group H lies in $\mathcal{F}$. Then for the Atiyah Conjecture to be true for extensions of finite or elementary amenable quotient of H, it is sufficient that the Atiyah Conjecture holds for specific subgroups of H.

So we do not have to verify the Atiyah Conjecture for H itself. However since H is a finite extension of itself, we see immediately that the result implies the Atiyah Conjecture for H.

Hence we can restate Theorem 4.59 of Linnell-Schick [28] in the following way:

THEOREM 7.5. *Let H be a discrete group which lies in $\mathcal{F}$, and let G be an arbitrary finite extension of H. Assume that KU fulfills the strong Atiyah Conjecture, where U is a subgroup of H such that for the inverse image G_S of an arbitrary Sylow-p subgroup G_S/H of G/H in G:*

- *U is normal in G_S;*
- *H/U is elementary amenable;*
- *$lcm(G_S/U) \mid lcm(G_S)$.*

Then the Atiyah Conjecture holds for KG as well.

Proof:

Let G_S/H be a Sylow-p subgroup of G/H and let G_S be its inverse image in G. If we can show that the Atiyah conjecture holds for every KG_S we are done (see [28, Lemma

2.4]).
Suppose U is a subgroup of H with the properties as mentioned above.
Since H has a finite classifying space, it is torsion free (the fundamental group of a finite space is torsion free) and so U, as a subgroup of H, is torsion free. Hence by [39, Lemma 3.4] U fulfills the Atiyah Conjecture, if and only if DU is a division ring. However by assumption U fulfills the Atiyah Conjecture and hence DU is a division ring and so is Artinian, since every division ring is Artinian.
Now we apply Prop. 2.6 of [28] to

$$1 \to U \to G_S \to G_S/U \to 1$$

First we will need Lemma 2.3 of [28]: We know that U fulfills the Atiyah Conjecture, hence we have:

$$\mathrm{lcm}(U)\mathrm{tr}_U(e) \in \mathbb{Z}$$

However U is torsion free and so $\mathrm{lcm}(U) = 1$ and we obtain

$$\mathrm{tr}_U(e) \in \mathbb{Z}$$

Moreover we know that DU is Artinian and $U \leq G_E$ with $|G_E/U|$ finite, so we can apply Lemma 2.3 of [28] with $L = \mathrm{lcm}(U) = 1$, i. e. we obtain:

$$|G_E/U|\mathrm{tr}_{G_E}(e) \in \mathbb{Z}$$

Now G_E/U is a finite subgroup of G_S/U and so $|G_E/U|$ divides $\mathrm{lcm}(G_S/U)$. Hence

$$\mathrm{lcm}(G_S/U)\mathrm{tr}_{G_E}(e) \in \mathbb{Z}$$

Now we can apply Proposition 2.6 of [28] to the above short exact sequence with $L = \mathrm{lcm}(G_S/U)$ (since L is independent of G_E and holds for all finite subgroups G_E/U of G_S/U.)
And so we obtain

$$\mathrm{lcm}(G_S/U)\mathrm{tr}_{G_S}(e) \in \mathbb{Z}$$

Using the divisibility relation we obtain

$$\mathrm{lcm}(G_S)\mathrm{tr}_{G_S}(e) \in \mathbb{Z}$$

as required. □
Note that there exist subgroups U of H with the required properties. However as a by-product of our construction, it turns out that H/U is torsion free. If U is a subgroup

of H that fulfills the strong Atiyah Conjecture and H/U is elementary amenable and torsion free, then by a result of Linnell, H itself fulfills the strong Atiyah Conjecture [39, Corollary 3.2]. In that case we would not need the result, because H itself fulfilled the strong Atiyah Conjecture and so the initial theorem of Linnell and Schick [28, Theorem 4.59] applies. It is not clear at the moment, whether there are subgroups U of H that have the required properties such that H/U is not torsion free.

Chapter 8

Appendix

As mentioned in Chapter 6 we will now demonstrate an extension of the proof of Hillman et al to obtain that abelian groups G with $cd(G) = 2$ that either admit a surjection onto $\mathbb{Z}$ or imbed injectively in their pro-p completions are cohomologically complete. Note that in both cases the group G admits surjections onto $\mathbb{Z}/p\mathbb{Z}$.
We included this extension in the appendix, because it is not clear, whether there is a large class of groups for which these requirements hold. Consider the following: Suppose G is abelian and $cd(G) = 2$.

- If G is finitely generated, then by the classification theorem of finitely generated abelian groups, $G = \mathbb{Z} \oplus \mathbb{Z}$.
- If G is countable, but not finitely generated, then G is a noncyclic subgroup of the additive group $\mathbb{Q}$. This is a result that has recently been obtained by Kropholler, Linnell and Lück [21]. These groups however do not admit a surjection onto $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$. So there are no countable abelian groups G with $cd(G) = 2$ that admit a surjection onto $\mathbb{Z}$ or imbed injectively in their pro-p completions.

Hence the groups satisfying our requirements must be uncountable.

8.1 Abelian Groups with surjection to $\mathbb{Z}$

We will take the advantage of considering groups of low cohomological dimension, that we explained in Section 6.2 on page 44.

THEOREM 8.1. *An abelian group G of cohomological dimension 2 is cohomologically complete, if it admits a surjective homomorphism $G \to \mathbb{Z}$.*

Notation: Let G be a group of cohomological dimension 2 that admits the above homomorphism. With $\gamma^i(G)$ we will denote the i-th component of the descending central series of G. Let M be a finite p-primary $\hat{G}^p$-module.
We shall show by induction on the length of M, i. e. the length of its finite composition series, that if H is an abelian subgroup of G of finite p-power index, then there is a smaller subgroup G_1 of G of finite p-power index such that the restriction is trivial:

$$\mathrm{res}^H_{G_1} : H^2(H, M) \to H^2(G_1, M)$$

Suppose first that $M = \mathbb{Z}/p\mathbb{Z}$ with trivial G-action:

$$\begin{array}{cccl} G & \times & \mathbb{Z}/p\mathbb{Z} & \to \mathbb{Z}/p\mathbb{Z} \\ (g & , & r) & \mapsto r \end{array}$$

By assumption there is a surjective homomorphism $t : G \to \mathbb{Z}$.
considering the restriction to H we obtain a surjective homomorphism

$$\tau : H \to \mathbb{Z}/p\mathbb{Z}$$

We can restrict t to H. Now $[t(G) : t(H)] < \infty$, since $[G : H] < \infty$:

> (suppose $h_1H, h_2H, \ldots h_sH$ are the conjugacy classes of H in G.
> I will show that $t(H)$ has only finitely many conjugacy classes in $t(G) = \mathbb{Z}$:
> For any element $x \in t(G)$ we have:
>
> $$\begin{aligned} xt(G) &= t(y)t(G) \text{ for some } y \in G \\ &= t(yG) \\ &= t(h_iG) \text{ for some } i = 1, \ldots s \end{aligned}$$
>
> Hence there are at most s conjugacy classes of $t(H)$ in $t(G)$, and so $[t(G) : t(H)] < \infty$. Thus $t(H)$ is a subgroup of $\mathbb{Z}^n$ of finite index, i.e. $t(H) \cong \mathbb{Z}$.)

Of course there exists a surjective homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ and considering the composition of this homomorphism and t restricted to H we obtain a surjective homomorphism

$$\tau : H \to \mathbb{Z}/p\mathbb{Z}$$

Thus $H/\ker(\tau) \cong \mathbb{Z}/p\mathbb{Z}$ and so $\ker(\tau)$ has index p in H. However by assumption $[G : H] = p^a$ for some a and hence:

$$[G : \ker(\tau)] = [G : H][H : \ker(\tau)] = p^a p = p^{a+1}$$

So $\ker(\tau)$ is a subgroup of G of finite p-power index. We will denote $\ker(\tau) = K$.
We have the following short exact sequence:

$$1 \to K \overset{i}{\to} H \overset{\pi}{\to} \mathbb{Z}/p\mathbb{Z} \to 1$$

So there exists a spectral sequence, the Lyndon/Hochschild-Serre spectral sequence, with initial term defined by $E_2^{s,t} = H^s(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})$. For further details on this spectral sequence consult for instance Weibel [41] or McCleary [33].
To see how the module map $\mathbb{Z}/p\mathbb{Z} \times H^t(K, \mathbb{Z}/p\mathbb{Z}) \to H^t(K, \mathbb{Z}/p\mathbb{Z})$ is defined, consider the following (more details can be found in Evens [13, p.72ff]):

Let
$$0 \overset{d_3}{\to} X_2 \overset{d_2}{\to} X_1 \overset{d_1}{\to} X_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0$$

be a projective $\mathbb{Z}H$-module resolution. Then it is also a projective $\mathbb{Z}K$-module resolution, since $\mathbb{Z}H$ is a free $\mathbb{Z}K$-module [13, p.35].

Now H acts on $\mathrm{Hom}_{\mathbb{Z}K}(X, \mathbb{Z}/p\mathbb{Z})$ by $(hf)(x) = h(f(h^{-1}x))$, where $\mathbb{Z}/p\mathbb{Z}$ is considered to be a $\mathbb{Z}H$-module. Then K acts trivially on $\mathrm{Hom}_{\mathbb{Z}K}(X, \mathbb{Z}/p\mathbb{Z})$, since we consider $\mathbb{Z}K$-module homomorphisms, and thus $(kf)(x) = k(f(k^{-1}x)) = kk^{-1}f(x) = f(x)$. Hence $\mathrm{Hom}_{\mathbb{Z}K}(X, \mathbb{Z}/p\mathbb{Z})$ can be considered to be a $\mathbb{Z}(H/K)$-module complex. This module map induces a module map on the cohomology groups (note: which is still well-defined!).

Start of Induction

The Lyndon/Hochschild-Serre spectral sequence for H as an extension of $\mathbb{Z}/p\mathbb{Z}$ by K has $\mathbb{E}_2$-term $E_2^{s,t} = H^s(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z}))$, r-th differential d_r of bidegree $(r, 1-r)$ and converges to $H^*(H, \mathbb{Z}/p\mathbb{Z})$. Note that for the higher terms we have the following recursive relation:

$$E_{r+1}^{s,t} \cong \ker d_r^{s,t} / \mathrm{im} d_r^{s-r,t+r-1} \tag{8.1}$$

Moreover we have the following:

$$E_2^{0,t} = H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) = H^t(K, \mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}/p\mathbb{Z}} \text{ and}$$

$$E_2^{s,0} = H^s(\mathbb{Z}/p\mathbb{Z}, H^0(K, \mathbb{Z}/p\mathbb{Z})) \overset{\star}{=} H^s(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$$

note that $H^0(K, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}^K = \{r \in \mathbb{Z}/p\mathbb{Z} \mid kr = r \forall k \in K\} = \mathbb{Z}/p\mathbb{Z}$, since we consider the trivial action, and therefore $\star$ holds.

Since $H^t(K, \mathbb{Z}/p\mathbb{Z}) = 0$ for $t > 2$ (cohomological dimension of G $\mathrm{cd}(G) \leq 2$ and since K is a subgroup of G we have $\mathrm{cd}(K) \leq 2$), there are only three nonzero rows, namely $H^s(\mathbb{Z}/p\mathbb{Z}, H^0(K, \mathbb{Z}/p\mathbb{Z}))$, $H^s(\mathbb{Z}/p\mathbb{Z}, H^1(K, \mathbb{Z}/p\mathbb{Z}))$ and $H^s(\mathbb{Z}/p\mathbb{Z}, H^2(K, \mathbb{Z}/p\mathbb{Z}))$.
I will show how the remaining terms of the spectral sequence look like:

$\mathbb{E}_3$:

$\ker d_2^{0,2}$	$\ker d_2^{1,2}$	$\ker d_2^{2,2}$	$\ker d_2^{3,2}$
$\ker d_2^{0,1}$	$\ker d_2^{1,1}$	$\ker d_2^{2,1}/\mathrm{im} d_2^{0,2}$	$\dots$
$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}/\mathrm{im} d_2^{0,1}$	$E_2^{3,0}/\mathrm{im} d_2^{1,1}$

$\mathbb{E}_4$:

$\ker d_3^{0,2}$	$\ker d_3^{1,2}$	$\ker d_3^{2,2}$	$\ker d_3^{3,2}$
$\ker d_2^{0,1}$	$\ker d_2^{1,1}$	$\ker d_2^{2,1}/\mathrm{im} d_2^{0,2}$	$\dots$
$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}/\mathrm{im} d_2^{0,1}$	$(E_2^{3,0}/\mathrm{im} d_2^{1,1})/\mathrm{im} d_3^{0,2}$

Now $d_4^{s,t} : E_4^{s,t} \to E_4^{s+4,t-3}$.
However if $t \in \{0, 1, 2\}$, then $t - 3 \notin \{0, 1, 2\}$ and so $E_4^{s+4,t-3} = 0$, and hence $d_4^{s,t} = 0$. Similarly, if $E_4^{s+4,t-3} \neq 0$, then $E_4^{s,t} = 0$ and due to equation (8.1) on the previous page we see that

$$E_4^{s,t} = E_5^{s,t} = E_6^{s,t} = \dots \tag{8.2}$$

Moreover applying the definition of $E_\infty^{s,t}$ (see for instance Ribes-Zalesskii [36])together with the fact that in our particular case $\mathrm{im} d_4^{s,t} = 0$ and $\ker d_4^{s,t} = E_4^{s,t}$ we obtain

$$E_\infty^{s,t} = E_4^{s,t} \tag{8.3}$$

And since $H^*(H, \mathbb{Z}/p\mathbb{Z}) = 0$ for $* > 2$ (because $\mathrm{cd}(G) \leq 2$ and $H \leq G$), we see that

$$d_3^{s,2} \text{ is an isomorphism for } s \geq 1. \tag{8.4}$$

The spectral sequence converges to $H^*(H, \mathbb{Z}/p\mathbb{Z})$. But this means that

$$E_\infty^{s,t} = F_{s+t}^{s}(H^{s+t}(H, \mathbb{Z}/p\mathbb{Z}))/F_{s+t}^{s+1}(H^{s+t}(H, \mathbb{Z}/p\mathbb{Z}))$$

where F_{s+t} is a filtration of $H^{s+t}(H, \mathbb{Z}/p\mathbb{Z})$.
However $\mathrm{cd}(H) \leq 2$ and so $H^{s+t}(H, \mathbb{Z}/p\mathbb{Z}) = 0$ for $s + t > 2$.
Therefore the filtration F_{s+t} must be 0 for $s + t > 2$.
Hence $E_\infty^{s,t} = 0$ for $s + t > 2$.
It follows $E_4^{s,t} = 0$ for $s + t > 2$ due to (8.3) on the preceding page.
So in particular $\ker d_3^{s,2} = 0$ for $s \geq 1$, which means $d_3^{s,2}$ is injective for $s \geq 1$. On the other hand from $E_4^{s,t} = 0$ for $s + t > 2$ we deduce $E_4^{s+3,0} = 0$ for $s \geq 1$. However

$$E_4^{s+3,0} = \ker d_3^{s+3,0}/\mathrm{im} d_3^{s,2} = (E_2^{s+3,0}/\mathrm{im} d_2^{s+1,1})/\mathrm{im} d_3^{s,2}$$

i. e. $(E_2^{s+3,0}/\mathrm{im} d_2^{s+1,1})/\mathrm{im} d_3^{s,2} = 0$, thus $E_2^{s+3,0}/\mathrm{im} d_2^{s+1,1} = \mathrm{im} d_3^{s,2}$ and so $d_3^{s,2}$ is surjective, and altogether $d_3^{s,2}$ is an isomorphism for $s \geq 1$.
Since $\mathbb{Z}/p\mathbb{Z}$ has cohomological period 2 we obtain isomorphisms

$$\gamma_2^{s,t} : E_2^{s,t} \cong E_2^{s+2,t}$$

such that

$$d_2^{s+2,t} \gamma_2^{s,t} = \gamma_2^{s+2,t-1} d_2^{s,t} \text{ for all } s \geq 1, t \geq 0. \tag{8.5}$$

We also obtain such isomorphisms for $s = 0, t \geq 0$:

$$\gamma_2^{0,t} : E_2^{0,t} \to E_2^{2,t}$$

$$\gamma_2^{0,t} : H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) \to H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z}))$$

Considering that

$$\begin{aligned} H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) &= (H^t(K, \mathbb{Z}/p\mathbb{Z}))^{\mathbb{Z}/p\mathbb{Z}} \\ H^n(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) &= (H^t(K, \mathbb{Z}/p\mathbb{Z}))^{\mathbb{Z}/p\mathbb{Z}}/NH^t(K, \mathbb{Z}/p\mathbb{Z}) \quad \text{for } n \text{ even }, n \geq 2 \end{aligned}$$

with $N = 1 + g + g^2 + \ldots + g^{p-1}$ where g is a generator of $\mathbb{Z}/p\mathbb{Z}$. However as K is central in H we know $NH^t(K, \mathbb{Z}/p\mathbb{Z}) = 0$, as we have explained in Section 6.3.2 on page 47, and hence

$$H^0(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) = H^2(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})).$$

Since the γ maps are isomorphisms, considering 8.5 on this page, we can conclude $d_2^{s+2,t} = d_2^{s,t}$.
Therefore we have isomorphisms $\ker d_2^{s,t} \cong \ker d_2^{s+2,t}$ and $\mathrm{im} d_2^{s,t} \cong \mathrm{im} d_2^{s+2,t}$.
In particular we have the isomorphisms

$$\gamma_3^{0,2} : E_3^{0,2} = \ker d_2^{0,2} \cong \ker d_2^{2,2} = E_3^{2,2} \text{ and}$$

$$\gamma_3^{3,0} : E_3^{3,0} = E_2^{3,0}/\mathrm{im}d_2^{1,1} \cong E_2^{5,0}/\mathrm{im}d_2^{3,1} = E_3^{5,0}$$

with

$$d_3^{2,2}\gamma_3^{0,2} = \gamma_3^{3,0}d_3^{0,2}. \tag{8.6}$$

It follows that $d_3^{0,2}$ is also an isomorphism, since all the other mappings involved in (8.6) are. Note that $d_3^{2,2}$ is an isomorphism, because of (8.4). Hence $E_\infty^{0,2} = 0$. But the edge homomorphism from $H^2(H, \mathbb{Z}/p\mathbb{Z})$ to $H^2(K, \mathbb{Z}/p\mathbb{Z})$ factors through

$$E_\infty^{0,2} \leq E_2^{0,2} = H^2(K, \mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}/p\mathbb{Z}}$$

and so is 0. The edge homomorphisms can be considered to be the following:

$$H^2(H, \mathbb{Z}/p\mathbb{Z}) \twoheadrightarrow E_\infty^{0,2} \subseteq E_r^{0,2} \subseteq \ldots \subseteq E_2^{0,2} = H^2(K)^{\mathbb{Z}/p\mathbb{Z}} \subseteq H^2(K, \mathbb{Z}/p\mathbb{Z})$$

Hence if $E_\infty^{0,2} = 0$ the whole edge homomorphism must be trivial. However the edge homomorphism is induced by the restriction map [41, 6.8.2, p.196] and so the restriction map $\mathrm{res}_K^H : H^2(H, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$ is the zero-map as well. Note that $E_4^{0,t} = E_5^{0,t} = \ldots = E_\infty^{0,t}$ as shown above, and $E_4^{0,t} = 0$ for $t \geq 3$ anyway, and hence for $t \geq 3$ the edge homomorphism $H^t(H, \mathbb{Z}/p\mathbb{Z}) \to H^t(K, \mathbb{Z}/p\mathbb{Z})$, which factors through $E_\infty^{0,t}$:

$$H^t(H, \mathbb{Z}/p\mathbb{Z}) \twoheadrightarrow E_\infty^{0,t} \subseteq E_r^{0,t} \subseteq \ldots \subseteq E_2^{0,t} = H^t(K)^{\mathbb{Z}/p\mathbb{Z}} \subseteq H^t(K, \mathbb{Z}/p\mathbb{Z})$$

is trivial anyway. Applying the same argument as above we can deduce that the restriction map $\mathrm{res}_K^H : H^t(H, \mathbb{Z}/p\mathbb{Z}) \to H^t(K, \mathbb{Z}/p\mathbb{Z})$ for $t \geq 3$ is the zero-map as well.

Induction step

In general M has a finite composition series.

If we consider $M = \mathbb{Z}/p\mathbb{Z}$ it has the following trivial composition series:

$$\{e\} \lhd \mathbb{Z}/p\mathbb{Z}$$

Suppose that M_1 is a maximal proper submodule of M with quotient $M/M_1 = \mathbb{Z}/p\mathbb{Z}$. Now for the induction step we assume the hypothesis is true for M_1 and show that it also holds for M.

Restriction res_K^H from H to K induces a homomorphism from the exact sequences of cohomology corresponding to the exact sequence of modules

$$0 \to M_1 \to M \to \mathbb{Z}/p\mathbb{Z} \to 0$$

So we obtain homomorphisms

$$\begin{CD}
H^2(H, M_1) @>{i^*}>> H^2(H, M) @>{\pi^*}>> H^2(H, \mathbb{Z}/p\mathbb{Z}) \\
@V{\mathrm{res}^H_K}VV @V{\mathrm{res}^H_K}VV @V{\mathrm{res}^H_K}VV \\
H^2(K, M_1) @>{i^*}>> H^2(K, M) @>{\pi^*}>> H^2(K, \mathbb{Z}/p\mathbb{Z})
\end{CD}$$

The result for $\mathbb{Z}/p\mathbb{Z}$ implies that the image of $H^2(H, M)$ lies in the image of $H^2(K, M_1)$. So the result for $\mathbb{Z}/p\mathbb{Z}$, that holds by the start of induction, means:

$$\mathrm{res}^H_K : H^2(H, \mathbb{Z}/p\mathbb{Z}) \to H^2(K, \mathbb{Z}/p\mathbb{Z})$$

is the zero map.
However the above diagram commutes and hence

$$\mathrm{res}^H_K(\pi^*(H^2(H, M))) = \pi^*(\mathrm{res}^H_K(H^2(H, M)))$$

So in particular $\mathrm{res}^H_K(H^2(H, M)) \subseteq \ker(\pi^*)$. But the rows are exact and hence

$$\mathrm{res}^H_K(H^2(H, M)) \subseteq \ker(\pi^*) = \mathrm{im}(i^*) = i^*(H^2(K, M_1)) \tag{8.7}$$

By the hypothesis of induction we may assume the result is true for M_1, and so there is a subgroup K_1 of finite p-power index of K such that restriction from $H^2(K, M_1)$ to $H^2(K_1, M_1)$ is trivial. We can extend the above diagram in the following way. (Of course it still commutes):

$$\begin{CD}
H^2(H, M_1) @>{i^*}>> H^2(H, M) @>{\pi^*}>> H^2(H, \mathbb{Z}/p\mathbb{Z}) \\
@V{\mathrm{res}^H_K}VV @V{\mathrm{res}^H_K}VV @V{\mathrm{res}^H_K}VV \\
H^2(K, M_1) @>{i^*}>> H^2(K, M) @>{\pi^*}>> H^2(K, \mathbb{Z}/p\mathbb{Z}) \\
@V{\mathrm{res}^K_{K_1}}VV @V{\mathrm{res}^K_{K_1}}VV @. \\
H^2(K_1, M_1) @>{i^*}>> H^2(K_1, M_1) @.
\end{CD}$$

By commutativity of the diagram we obtain

$$\mathrm{res}^K_{K_1} i^* = i^* \mathrm{res}^K_{K_1}$$

and by induction hypothesis $\mathrm{res}^K_{K_1} : H^2(K, M_1) \to H^2(K_1, M_1)$ is trivial.
Hence $i^*\mathrm{res}^K_{K_1}$ must be trivial as well.
However in (8.7) we have shown that $\mathrm{res}^H_K(H^2(H, M)) \subseteq i^*(H^2(K, M_1))$ and so

$$\mathrm{res}^K_{K_1}\mathrm{res}^H_K(H^2(H, M)) = 0$$

But $\mathrm{res}^K_{K_1}\mathrm{res}^H_K = \mathrm{res}^H_{K_1}$
and hence restriction from $H^2(H, M)$ to $H^2(K_1, M)$ is also trivial. This establishes the inductive step.
In particular restriction from $H^2(G, \mathbb{Z}/p\mathbb{Z})$ to $H^2(J, \mathbb{Z}/p\mathbb{Z})$ is trivial for some subgroup J of finite p-power index, and so the result follows, as in [40, Ex. 1, Chapter 1, Section 2.6].

THEOREM 8.2. *Let G be a discrete group and let $j : G \to \hat{G}^p$ be the canonical map from G into its pro-p-completion. Let M be an arbitrary finite $\hat{G}^p$-module. Then the following are equivalent:*

$\mathbf{A_n}$: *$H^q(\hat{G}^p, M) \to H^q(G, M)$ is bijective for all M for $q \leq n$ and injective for $q = n + 1$.*

$\mathbf{B_n}$: *For all $x \in H^q(G, M)$ with $1 \leq q \leq n$, there exists a subgroup G_0 of G, the inverse image of a subgroup of $\hat{G}^p$ of finite p-power index, such that x is mapped to 0 in $H^q(G_0, M)$.*

Moreover A_0 always holds and in the case of the profinite group to be the pro-p-completion of G, A_1 holds as well.

However we have shown that $\mathrm{res}^G_J : H^q(G, \mathbb{Z}/p\mathbb{Z}) \to H^q(J, \mathbb{Z}/p\mathbb{Z})$ is trivial for $q = 2$. Moreover for trivial reasons, since $cd(G) = 2$, it is the zero map for $q > 2$, and so altogether $\mathbf{B_n}$ holds for $n \geq 0$. Finally this means that $\mathbf{A_n}$ holds for all $n \geq 0$. And so G is cohomologically complete (p-good). □
Considering the main theorem of this section we deduce

COROLLARY 8.3. *Let G be an abelian group with $cd(G) = 2$. If G admits a surjection onto the integers, then $cd(\hat{G}^p) = 2$.*

8.2 Abelian Groups that imbed in their pro-p-completions

One important property, Hillman, Matei and Morishita need in their proof is the existence of surjections $H \to \mathbb{Z}/p\mathbb{Z}$ for subgroups H of finite p-power index for all primes p. Link groups have this property and so they could assume it. In the previous section, instead of considering link groups, we considered abelian groups that admit a surjection onto the integers, and so we also obtain the necessary surjections. Now we consider abelian groups that imbed injectively in every pro-p-completion and obtain:

THEOREM 8.4. *Let G be an abelian group with $cd(G) = 2$ that imbeds injectively in its pro-p-completion for every prime p, then G is cohomologically complete.*

Proof:
Assume $j_p : G \to \hat{G}^p$ is injective for every prime p. Now $\ker j_p = \bigcap_{H \in I_p} H = \{e\}$, where $I_p = \{H \lhd G \mid H \text{ has finite } p - \text{power index in } G\}$. So in particular $\{G\} \subsetneq I_p$. So there exists a normal subgroup H of G, s. t. G/H is a p-group. Assume $|G/H| = p^n$. Then G/H has a subgroup of order p^l for $l = 0, \ldots n$. Thus G/H has a subgroup G_1/H of order p^{n-1}. Now

$$(G/H)/(G_1/H) \cong G/G_1$$

and so G_1 is a subgroup of index p of G, and so $G/G_1 \cong \mathbb{Z}/p\mathbb{Z}$ and G admits a surjection onto $\mathbb{Z}/p\mathbb{Z}$.
Moreover due to the universal property of $\hat{H}^p$ the following diagram commutes:

$$\begin{array}{ccc} H & \xhookrightarrow{i} & G \\ \big\downarrow{\scriptstyle j_H} & & \big\downarrow{\scriptstyle j_G} \\ \hat{H}^p & \longrightarrow & \hat{G}^p \end{array}$$

Hence $H \to \hat{H}^p$ is injective as well, and so by the same argument as above, H admits a surjection onto $\mathbb{Z}/p\mathbb{Z}$. We denote the kernel of such a surjection by K.
So we obtain a short exact sequence

$$1 \to K \to H \to \mathbb{Z}/p\mathbb{Z} \to 1$$

thus there exists a spectral sequence, the Lyndon/Hochschild-Serre spectral sequence,

$$E_2^{s,t} = H^s(\mathbb{Z}/p\mathbb{Z}, H^t(K, \mathbb{Z}/p\mathbb{Z})) \Rightarrow H^\star(H, \mathbb{Z}/p\mathbb{Z})$$

From here onwards, the proof is exactly the same as in the previous section, where we considered abelian groups with cohomological dimension 2 and surjection onto the integers.
Hence abelian groups G with cohomological dimension 2, that imbed injectively in their pro-p-completion for every prime p are cohomologically complete. □
Analogously to the previous section we obtain:

COROLLARY 8.5. *Let G be an abelian group with $cd(G) = 2$. If G imbeds injectively in its pro-p-completion for every prime p, then $cd(\hat{G}^p) = 2$.*

Bibliography

[1] D. J. Anick. Non-commutative graded algebras and their Hilbert series. *Journal of Algebra*, 78:120–140, 1982.

[2] D. J. Anick. Inert sets and the lie algebra associated to a group. *Journal of Algebra*, 111:154–165, 1987.

[3] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque*, 32–33:43–72, 1976.

[4] G. Baumslag. Positive one-relator groups. *Trans. Amer. Math. Soc.*, 156:165–183, 1971.

[5] G. Baumslag, E. Dyer, and A. Heller. The topology of discrete groups. *J. Pure and Applied Algebra*, 16:1–47, 1980.

[6] N. Bourbaki. *Algebra*, volume 1236 of *Actualités scientifiques et industrielles*. Hermann, Paris, 1960.

[7] N. Bourbaki. *Groupes et Algebres de Lie*, volume 1285 of *Actualités scientifiques et industrielles*. Hermann, Paris, 1960.

[8] S. Boyer, D. Rolfsen, and B. Wiest. Orderable 3-manifold groups. *Ann. Inst. Fourier, Grenoble*, 55(1):243–288, 2005.

[9] G. E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1997.

[10] G. H. Cliff. Zero divisors and idempotents in group rings. *Canadian J. Math.*, 32:596–602, 1980.

[11] J. M. Cohen. Calculations on the unique product property. *University of Pennsylvania*.

[12] J. M. Cohen. Zero divisors in group rings. *Comm. in Algebra*, 2(1):1–14, 1974.

[13] L. Evens. *The Cohomology of Groups*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 1991.

[14] D. R. Farkas and R. L. Snider. k_0 and noetherian group rings. *Journal of Algebra*, 42:192–198, 1976.

[15] E. Formanek. The zero divisor question for supersolvable groups. *Bull. Austral. Math. Soc.*, 9:67–71, 1973.

[16] G. Higman. The units of group rings. *Proc. London Math. Soc.*, 46(2):231–248, 1940.

[17] J. Hillman, D. Matei, and M. Morishita. Pro-p link groups and p-homology groups. In *Primes and Knots, Contemporary Mathematics*, pages 121 –136. AMS, Providence, R. I., 2006.

[18] J. Howie and H. Short. The band-sum problem. *J. London Math. Soc.*, 31:571–576, 1985.

[19] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, volume 9 of *Graduate Texts in Mathematics*. Springer, New York, 1972.

[20] H. Koch. Über pro-p-Gruppen der kohomologischen Dimension 2. *Math. Nachr.*, 78:285–289, 1977.

[21] P. Kropholler, P. Linnell, and W. Lück. Groups of small homological dimension and the Atiyah Conjecture. *Preprint*, pages 1–6, 2004.

[22] J. P. Labute. Algèbres de Lie et pro-p groupes définis par une seule relation. *Inventiones math.*, 4:142 –158, 1967.

[23] J. P. Labute. On the descending central series of groups with a single defining relation. *J. Algebra*, 14:16–23, 1970.

[24] J. P. Labute. The determination of the Lie algebra associated to the lower central series of a group. *Trans. Amer. Math. Soc.*, 288(1):51–57, 1985.

[25] J. P. Labute. The Lie algebra associated to the lower central series of a link group and Murasugi's conjecture. *Proc. Amer. Math. Soc.*, 109(4):951–956, 1990.

[26] J. P. Labute. Mild pro-p-groups and Galois groups of p-extensions of $\mathbb{Q}$. *J. Reine u. Angew. Mathematik*, 596, 2006.

[27] R. H. LaGrange and A. H. Rhemtulla. A remark on the group rings of order preserving permutation groups. *Can. Math. Bull.*, 11:679–680, 1968.

[28] P. Linnell and T. Schick. Finite Group Extensions and the Atiyah Conjecture. *To appear in Journal of the AMS*, 2007.

[29] P. Linnell and T. Schick. Galois cohomology of completed link groups. *Preprint*, pages 1–11, 2007.

[30] W. Lück. L^2-Invariants of regular coverings of compact manifolds and CW-complexes. In *Handbook of Geometric Topology*, pages 735–817. Elsevier, Amsterdam, 2002.

[31] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 2002.

[32] R. C. Lyndon. Cohomology theory of groups with a single defining relation. *Annals of Math.*, 52(3):650–665, 1950.

[33] J. McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[34] D. S. Passman. Advances in group rings. *Israel J. Math.*, 19:67–107, 1974.

[35] D. S. Passman. *The algebraic structure of group rings*. Pure and Applied Mathematics. John Wiley and Sons, New York, 1977.

[36] L. Ribes and P. Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 2000.

[37] D. Rolfsen. *Knots and Links*. Publish or Perish, Houston, second edition, 1990.

[38] J. J. Rotman. *An Introduction to Homological Algebra*. Pure and Applied Mathematics. Academic Press, Inc., San Diego, 1979.

[39] T. Schick. Integrality of L^2-Betti numbers. *Math. Ann.*, 317(4):727 – 750, 2000.

[40] J.-P. Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[41] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge studies in advanced mathematics*. Cambridge University Press, New York, 1994.

[42] J. S. Wilson. *Profinite groups*, volume 19 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1998.

Curriculum Vitae

Persönliche Daten Inga Christine Blomer, geborene Kümpel
geboren am 4. Dezember 1979 in Kiel
als zweites Kind von Hans-Joachim Kümpel und Monika Kümpel

Familienstand verheiratet, ein Kind

Bildung

Schule

08/91 – 07/93	Gymnasium Kronshagen bei Kiel
08/93 – 06/99	Jugenddorf-Christophorusschule Königswinter bei Bonn, 12. Juni 1999 Abitur

Universität

10/99 – 09/01	Universität Trier 11. September 2001 Vordiplom in Mathematik
10/01 – 07/03	University of Southampton, England 04. Juli 2003 Master of Mathematics
seit 10/03	Universität Göttingen Stipendiatin des DFG-Graduiertenkollegs "Gruppen und Geometrie" 09/05 – 12/05 Université de Nice, Frankreich; Forschungsaufenthalt gefördert vom DAAD 09/06 – 01/07 Babypause

Hiermit versichere ich an Eides statt, dass ich die Dissertation selbständig und ohne unerlaubte Hilfe angefertigt habe.

Göttingen, den 1. Juni 2007